THE

ELEMENTS OF EUCLID.

THE ELEMENTS OF EUCLID;

VIZ.

THE FIRST SIX BOOKS,

TOGETHER WITH THE

ELEVENTH AND TWELFTH.

PRINTED,

WITH A FEW VARIATIONS AND IMPROVEMENTS, FROM THE TEXT OF DR. SIMSON.

WITH

AN APPENDIX,

CONTAINING

MISCELLANEOUS EXERCISES IN PLANE GEOMETRY, AND CRITICAL QUESTIONS AND NOTES ON THE FIRST SIX BOOKS.

A NEW EDITION, CORRECTED AND REVISED.

BY WILLIAM RUTHERFORD, F. R. A. S.

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THE EDITOR'S PREFACE.

In this edition of the Elements of Euclid, the text and arrangement of Dr. Simson have been strictly adhered to, with the exception of several corrections and modifications, where the language of that distinguished geometer is not in strict accordance with geometrical accuracy.

In many instances Euclid employs an *indirect* method for establishing the truth of what he intends to prove, and deduces an inference which contradicts some *self-evident*, or *demonstrated* truth. Whenever an inference is contrary to a demonstrated truth, the language of Simson is frequently ambiguous, and does not clearly point out whether the inference or the demonstrated truth is impossible. In the present edition this ambiguity has been removed by a slight change of the language, and in every instance the inference has been stated to be inconsistent with the demonstrated truth.

Numerous instances, too, occur in the Elements, where the demonstrations might be improved. For instance, in the demonstration of the Twenty-third Proposition of the First Book, it is proved that the *given* rectilineal angle is equal to the rectilineal angle *required* to be made; but this is precisely the reverse of what should have been demonstrated; because the angle which is made should be proved equal to the angle which is given, and the demonstration is equally simple in either way. In one or two cases of this kind, the necessary modification of the demonstration has been attended to, whilst in others the demonstrations have been allowed to remain, in deference to the general usage of geometers; but they may easily be demonstrated by the student, in strict conformity with the language of the enunciation of the Proposition.

The demonstration of the First Proposition of the Third Book has been long known to be defective, and it is here rendered valid by the addition of a line or two to the original text. Besides, wherever a looseness of the phraseology of Simson exists, the Editor has endeavoured to supply the defect by a slight alteration of the language.

To the Elements have been added an Appendix, containing a selection of Propositions for exercise on the first six Books; besides a number of critical questions on *Euclid,* which, it is hoped, will be found useful in pro-

moting accuracy and perspicuity, and in improving the active investigating powers of intellect.

With a view to greater clearness, and to arrest the attention of the student at every step of the reasoning, marginal references have been added where they appeared to be wanting; and, as utility rather than novelty has been the Editor's principal object throughout the work, it is presumed that the changes which have been made in this edition will be found to be real improvements.

WILLIAM RUTHERFORD.

Royal Military Academy,
Woolwich, March, 1841.

THE

ELEMENTS OF EUCLID.

BOOK I.

DEFINITIONS.

I.

A *point* is that which hath no parts, or which hath no magnitude.

II.

A *line* is length without breadth.

III.

The *extremities of a line* are points.

IV.

A *straight line* is that which lies evenly between its extreme points.

V.

A *superficies* is that which hath only length and breadth.

VI.

The *extremities of a superficies* are lines.

VII.

A *plane superficies* is that in which any two points being taken, the straight line between them lies wholly in that superficies.

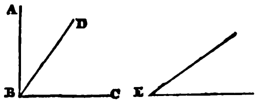
VIII.

"A *plane angle* is the inclination of two lines to one

"another in a plane, which meet together, but are
"not in the same direction."

IX.

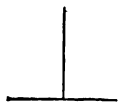
A *plane rectilineal angle* is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.



N. B. 'When several angles are at one point B, any 'one of them is expressed by three letters, of which 'the letter that is at the vertex of the angle, that is, 'at the point in which the straight lines that contain 'the angle meet one another, is put between the other 'two letters, and one of these two is somewhere upon 'one of those straight lines, and the other upon the 'other line: thus the angle which is contained by the 'straight lines AB, CB, is named the angle ABC, or 'CBA; that which is contained by AB, BD, is named 'the angle ABD, or DBA; and that which is con- 'tained by BD, CB, is called the angle DBC, or CBD; 'but, if there be only one angle at a point, it may be 'expressed by a letter placed at that point; as the 'angle at E.'

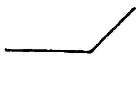
X.

When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a *right angle;* and the straight line which stands on the other is called a *perpendicular* to it.



XI.

An *obtuse angle* is that which is *greater* than a right angle.



XII.

An *acute angle* is that which is less than a right angle.

XIII.

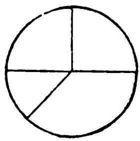
"A *term* or *boundary* is the extremity of anything."

XIV.

A *figure* is that which is enclosed by one or more boundaries.

XV.

A *circle* is a plane figure contained by one line, which is called the *circumference*, and is such that all straight lines drawn from a certain point within the figure to the circumference, are equal to one another.



XVI.

And this point is called the *centre* of the circle.

XVII.

A *diameter* of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

XVIII.

A *semicircle* is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XIX.

"A *segment of a circle* is the figure contained by a straight line, and *the* circumference it cuts off."

XX.

Rectilineal figures are those which are contained by straight lines.

XXI.

Trilateral figures, or *triangles*, by three straight lines.

XXII.

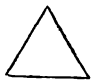
Quadrilateral figures, by four straight lines.

XXIII.

Multilateral figures, or *polygons*, by more than four straight lines.

XXIV.

Of three-sided figures, an *equilateral triangle* is that which has three equal sides.

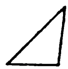


XXV.

An *isosceles triangle* is that which has only two sides equal.

XXVI.

A *scalene triangle* is that which has three unequal sides.



XXVII.

A *right-angled triangle* is that which has a right angle.

XXVIII.

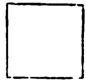
An *obtuse angled triangle* is that which has an obtuse angle.

XXIX.

An *acute-angled triangle* is that which has three acute angles.

XXX.

Of four-sided figures, a ***square*** is that which has all its sides equal, and all its angles right angles.

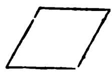


XXXI.

An ***oblong***, or ***rectangle***, is that which has all its angles right angles, but has not all its sides equal.

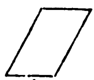
XXXII.

A ***rhombus*** is that which has all its sides equal, but its angles are not right angles.



XXXIII.

A ***rhomboid*** is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.



XXXIV.

Any other four-sided figure, besides these, is called a ***trapezium***.

XXXV.

Parallel straight lines are such as are in the same plane, and which, however far produced either way, do not meet.

POSTULATES.

I.

Let it be granted, that a straight line may be drawn from any one point to any other point.

II.

That a terminated straight line may be produced to any length in a straight line.

III.

And that a circle may be described from any centre, at any distance from that centre.

AXIOMS.

I.

Things which are equal to the same thing, are equal to one another.

II.

If equals be added to equals, the wholes are equal.

III.

If equals be taken from equals, the remainders are equal.

IV.

If equals be added to unequals, the wholes are unequal.

V.

If equals be taken from unequals, the remainders are unequal.

VI.

Things which are double of the same, are equal to one another.

VII.

Things which are halves of the same, are equal to one another.

VIII.

Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

IX.

The whole is greater than its part.

X.

Two straight lines cannot enclose a space.

XI.

All right angles are equal to one another.

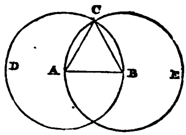
XII.

"If a straight line meet two straight lines, which are
" in the same plane, so as to make the two interior
" angles on the same side of it, taken together, less
" than two right angles, these straight lines being
" produced, shall at length meet upon that side on
" which are the angles which are less than two
" right angles."

PROPOSITION I. PROBLEM.

To describe an equilateral triangle upon a given finite straight line.

Let AB be the given straight line; it is required to describe an equilateral triangle upon it.



From the centre A, at the distance AB, describe* the circle BCD, and from the centre B, at the distance BA, describe the circle ACE; and from the point C, in which the circles cut one another, draw the straight lines* CA, CB, to the points A, B; ABC shall be an equilateral triangle.

* 3 Postulate.

* 1 Post.

Because the point A is the centre of the circle BCD, AC is equal* to AB; and because the point B is the centre of the circle ACE, BC is equal to BA: but it has been proved that CA is equal to AB; therefore CA, CB, are each of them equal to AB; but things which are equal to the same thing are equal* to one another; therefore CA is equal to CB; wherefore CA, AB, BC, are equal to one another; and the triangle ABC is therefore equilateral, and it is described upon the given straight line AB. Which was required to be done.

* 15 Definition.

* 1st Axiom.

PROP. II. PROB.

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line; it is required to draw from the point A a straight line equal to BC.

From the point A to B draw* the straight line AB; and upon it describe* the equilateral triangle DAB, and produce* the straight line DB to E; from the centre B, at the distance BC, describe* the circle *CFH, and from* the centre D, at the distance DF,

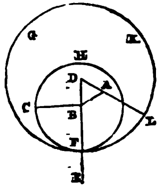
* 1 Post.

* 1. 1.

* 2 Post.

* 3 Post.

describe the circle FGK: produce the straight line DA to meet the circle FGK in L; AL shall be equal to BC.

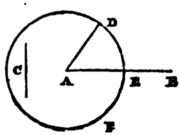


Because the point B is the centre of the circle CFH, BC is equal* to BF; and because D is the centre of the circle FGK, DL is equal to DF, and DA, DB, parts of them, are equal; therefore the remainder AL is equal* to the remainder BF: but it has been shown, that BC is equal to BF; wherefore AL and BC are each of them equal to BF; and things that are equal to the same thing are equal* to one another; therefore the straight line AL is equal to BC. Wherefore, from the given point A, a straight line AL has been drawn equal to the given straight line BC. Which was to be done.

* 15 Def.
* 3 Ax.
* 1 Ax.

PROP. III. PROB.

From the greater of two given straight lines to cut off a part equal to the less.



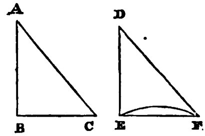
Let AB and C be the two given straight lines, whereof AB is the greater. It is required to cut off from AB, the greater, a part equal to C, the less.

From the point A draw* the straight line AD equal to C; and from the centre A, and at the distance AD, describe* the circle DEF: then AE shall be equal to C. Because A is the centre of the circle DEF, AE is equal* to AD; but the straight line C is likewise equal to AD; whence AE and C are each of them equal to AD; wherefore the straight line AE is equal* to C, and therefore from AB, the greater of two straight lines, a part AE has been cut off equal to C the less. Which was to be done.

* 2. 1.
* 3 Post.
* 15 Def.
* 1 Ax.

PROP. IV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each; and have likewise the angles contained by those sides equal to one another; they shall likewise have their bases, or third sides, *equal; and the two triangles shall be equal; and their other angles shall be equal, each to each, viz. those to which the equal sides are opposite.*



Let ABC, DEF be two triangles, which have the two sides AB, AC equal to the two sides DE, DF, each to each, viz. AB to DE, and AC to DF; and the angle BAC equal to the angle EDF, the base BC shall be equal to the base EF; and the triangle ABC to the triangle DEF; and the other angles to which the equal sides are opposite, shall be equal, each to each, viz. the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

For, if the triangle ABC be applied to DEF, so that the point A may be on D, and the straight line AB upon DE; the point B shall coincide with the point E, because AB is equal to DE; and AB coinciding with DE, AC shall coincide with DF, because the angle BAC is equal to the angle EDF: wherefore, also, the point C shall coincide with the point F, because the straight line AC is equal to DF: but the point B was proved to coincide with the point E; wherefore the base BC shall coincide with the base EF; because the point B coinciding with E, and C with F, if the base BC did not coincide with the base EF, two straight
* 10 Ax. lines would enclose a space, which is impossible.* Therefore the base BC coincides with the base EF, and is therefore equal to it. Wherefore the whole triangle ABC coincides with the whole triangle DEF,

and is equal to it; and the other angles of the one coincide with the remaining angles of the other, and are equal to them, viz. the angle ABC to the angle DEF, and the angle ACB to DFE. Therefore, if two triangles have two sides of the one equal to two sides of the other, each to each, and have, likewise, the angles contained by those sides equal to one another, their bases shall likewise be equal, and the triangles shall be equal, and their other angles, to which the equal sides are opposite, shall be equal, each to each. Which was to be demonstrated.

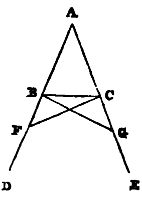
PROP. V. THEOR.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall be equal.

Let ABC be an isosceles triangle, of which the side AB is equal to AC, and let the straight lines AB, AC, be produced to D and E: the angle ABC shall be equal to the angle ACB, and the angle CBD to the angle BCE.

In BD take any point F, and from AE the greater, cut off AG equal* to AF, the less, and join FC, GB. * 3. 1.

Because AF is equal to AG, and AB to AC, the two sides FA, AC, are equal to the two GA, AB, each to each; and they contain the angle FAG common to the two triangles AFC, AGB; therefore the base FC is equal* to the base GB, and the triangle AFC to the triangle AGB; and the remaining angles of the one are equal* to the remaining angles of the other, each to each, to which the equal sides are opposite; viz. the angle



* 4. 1.

* 4. 1.

ACF to the angle ABG, and the angle AFC to the angle AGB: and because the whole AF is equal to the whole AG, of which the parts AB, AC, are equal; the
* 3 Ax. remainder BF is equal* to the remainder CG; and FC was proved to be equal to GB; therefore the two sides BF, FC are equal to the two CG, GB, each to each; and the angle BFC has been proved to be equal to the angle CGB; wherefore the two triangles BFC,
* 4. 1. CGB are equal,* and their remaining angles are equal, each to each, to which the equal sides are opposite; therefore the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG. And, since it has been demonstrated, that the whole angle ABG is equal to the whole ACF, the parts of which, the angles CBG, BCF are also equal; the remaining angle ABC is therefore equal to the remaining angle ACB, which are the angles at the base of the triangle ABC: and it has also been proved that the angle FBC is equal to the angle GCB, which are the angles upon the other side of the base. Therefore the angles at the base, &c. Q. E. D.

COROLLARY. Hence every equilateral triangle is also equiangular.

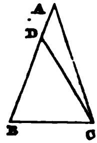
PROP. VI. THEOR.

If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, *the equal angles, shall be equal to one another.*

Let ABC be a triangle having the angle ABC equal to the angle ACB; the side AB shall be equal to the side AC.

For, if AB be not equal to AC, one of them is greater than the other; let AB be the greater, and
* 3. 1. from it cut off* DB equal to AC, the less, and join DC; therefore, because in the triangles DBC, ACB, DB is equal to AC, and BC common to both, the two

sides DB, BC are equal to the two AC, CB, each to each; and the angle DBC is equal to the angle ACB; therefore the base DC is equal to the base AB, and the triangle DBC is equal* to the triangle ACB, the less to the greater, which is absurd. Therefore AB is not unequal to AC, that is, it is equal to it. Wherefore, if two angles, &c. Q. E. D. * 4. 1

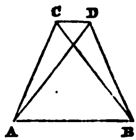


COR.—Hence every equiangular triangle is also equilateral.

PROP. VII. THEOR.

Upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise *those which are terminated in the other extremity, equal to one another.*

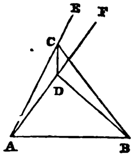
If it be possible, upon the same base AB, and on the same side of it, let ACB, ADB be two triangles, which have their sides CA, DA terminated in A, one extremity of the base, equal to one another, and likewise their sides CB, DB, terminated in B, the other extremity, equal to one another.



Join CD; then, in the case in which the vertex of each of the triangles is without the other triangle, because AC is equal to AD, the angle ACD is equal* to the angle ADC; but the angle ACD is greater than the angle BCD; therefore the angle ADC is also greater than the angle BCD; much more then is the angle BDC greater than the angle BCD. Again, because CB is equal to DB, by hypothesis, the angle BDC is equal* to the angle BCD; but this is impossible, because it has been proved that the angle BDC is greater than the angle BCD. * 5. 1. * 5. 1.

But if one of the vertices, as D, be within the other

triangle ACB; produce AC, AD to E, F; therefore, because in the triangle ACD, AC is equal to AD, the angles ECD, FDC upon the other side of the base CD are equal* to one another; but the angle ECD is greater than the angle BCD; therefore the angle FDC is likewise greater than BCD; much more then is the angle BDC greater than the angle BCD. Again, because CB is equal to DB, the angle BDC is equal* to the angle BCD; but this is impossible, because the angle BDC has been proved to be greater than the same angle BCD. The case in which the vertex of one triangle is upon a side of the other, needs no demonstration.



* 5. 1.

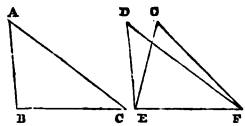
* 5. 1.

Therefore, upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated in the other extremity. Q. E. D.

PROP. VIII. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them, of the other.

Let ABC, DEF be two triangles, having the two sides AB, AC equal to the two sides DE, DF, each to each, viz. AB to DE, and AC to DF; and also the base BC equal to the base EF; the angle BAC shall be equal to the angle EDF.



For, if the triangle ABC be applied to DEF, so

that the point B be on E, and the straight line BC upon EF; the point C shall also coincide with the point F, because BC is equal to EF. Therefore BC coinciding with EF, BA and AC shall coincide with ED and DF; for, if the base BC coincide with the base EF, but the sides BA, CA do not coincide with the sides ED, FD, but have a different situation as EG, FG; then, upon the same base EF, and upon the same side of it, there can be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise their sides terminated in the other extremity: but this is impossible;* therefore, if the base BC coincide with the base EF, the sides BA, AC cannot but coincide with the sides ED, DF; wherefore likewise the angle BAC coincides with the angle EDF, and is equal* to it. Therefore if two triangles, &c. Q. E. D.

* 7. 1.

* 8 Ax.

PROP. IX. PROB.

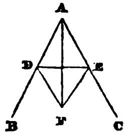
To bisect a given rectilineal angle, that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle, it is required to bisect it.

Take any point D in AB, and from AC cut off* AE equal to AD; join DE, and upon it describe,* on the side remote from A, an equilateral triangle DEF; then join AF: the straight line AF shall bisect the angle BAC.

* 3. 1.

* 1. 1.



Because AD is equal to AE, and AF is common to the two triangles DAF, EAF; the two sides DA, AF, are equal to the two sides EA, AF, each to each; and the base DF is equal to the base EF; therefore the angle DAF is

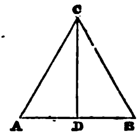
* 8. 1. equal* to the angle EAF; wherefore the given rectilineal angle BAC is bisected by the straight line AF. Which was to be done.

PROP. X. PROB.

To bisect a given finite straight line, that is, to divide it into two equal parts.

Let AB be the given straight line; it is required to divide it into two equal parts.

* 1. 1. Describe* upon it an equilateral triangle ABC, and
* 9. 1. bisect* the angle ACB by the straight line CD; then AB shall be cut into two equal parts in the point D.



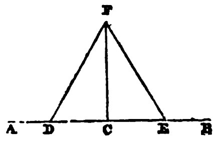
Because AC is equal to CB, and CD common to the two triangles ACD, BCD; the two sides AC, CD are equal to BC, CD, each to each; and the angle ACD is equal to the angle BCD; therefore the base AD
* 4. 1. is equal* to the base DB, and the straight line AB is divided into two equal parts in the point D. Which was to be done.

PROP. XI. PROB.

To draw a straight line at right angles to a given straight line, from a given point in the same.

Let AB be a given straight line, and C a point given in it; it is required to draw from the point C a straight line at right angles to AB.

* 3. 1. Take any point D in AC, and* make CE equal to
* 1. 1. CD, and upon DE describe* the equilateral triangle DFE, and join FC; the straight line FC drawn from the given point C shall be at right angles to the given straight line AB.

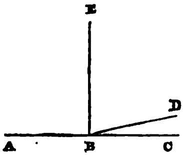


Because DC is equal to CE, and FC common to the two triangles DCF, ECF;

the two sides DC, CF are equal to the two EC, CF, each to each; and the base DF is equal to the base EF; therefore the angle DCF is equal* to the angle * 8. 1.
ECF; and they are adjacent angles. But, when the adjacent angles which one straight line makes with another straight line are equal to one another, each of them is called a right* angle; therefore each of the *10 De
angles DCF, ECF, is a right angle. Therefore, from the given point C, in the given straight line AB, FC has been drawn at right angles to AB. Which was to be done.

Cor. By help of this problem, it may be demonstrated, that two straight lines cannot have a common segment.

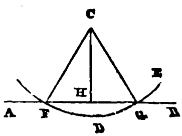
If it be possible, let the two straight lines ABC, ABD have the segment AB common to both of them. From the point B draw BE at right angles to AB; and because ABC is a straight line, the angle CBE is equal* to the *10 De
angle EBA; in the same manner, because ABD is a straight line, the angle DBE is equal to the angle EBA; wherefore the angle DBE is equal to the angle CBE, the less to the greater; which is impossible: therefore two straight lines cannot have a common segment.



PROP. XII. PROB.

To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

Let AB be the given straight line, which may be produced any length both ways, and let C be a point without it. It is required to draw from the point C a *straight line perpendicular to AB.*



Take any point D upon the other side of AB, and
* 3 Post. from the centre C, at the distance CD, describe* the
* 10. 1. circle EGF meeting AB in F, G; and bisect* FG in H, and join CH; the straight line CH, drawn from the given point C, shall be perpendicular to the given straight line AB.

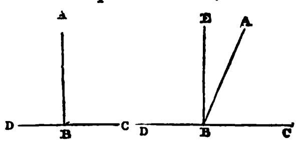
Join CF, CG, and because FH is equal to HG, and HC common to the two triangles FHC, GHC, the two sides FH, HC are equal to the two GH, HC, each to
* 15 Def. each; and the base CF is equal* to the base CG;
* 8. 1. therefore the angle CHF is equal* to the angle CHG; and they are adjacent angles; but when a straight line standing on another straight line makes the adjacent angles equal to one another, each of them is a right angle; and the straight line which stands upon the other is called a perpendicular to it; therefore from the given point C a perpendicular CH has been drawn to the given straight line AB. Which was to be done.

PROP. XIII. THEOR.

The angles which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.

Let the straight line AB make with CD, upon one side of it, the angles CBA, ABD; these shall be either two right angles, or shall be together equal to two right angles.

For if the angle CBA be equal to ABD, each of
* 10 Def. them is a right* angle: but if not, from the point B draw* BE at right
* 11. 1. angles to CD; therefore the angles



* 10 Def. CBE, EBD are two right angles;* and because the angle CBE is equal to the two angles CBA, ABE, add

the angle EBD to each of these equals; therefore the angles CBE, EBD are equal* to the three angles CBA, ABE, EBD. Again, because the angle DBA is equal to the two angles DBE, EBA, add the angle ABC to each of these equals; therefore the angles DBA, ABC are equal to the three angles DBE, EBA, ABC; but the angles CBE, EBD have been demonstrated to be equal to the same three angles; and things that are equal to the same thing are equal* to one another; therefore the angles CBE, EBD are equal to the angles DBA, ABC; but CBE, EBD are two right angles; therefore DBA, ABC are together equal to two right angles. Wherefore, the angles, &c. Q. E. D.

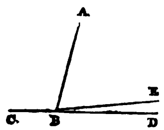
* 2 Ax.

* 1 Ax.

PROP. XIV. THEOR.

If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

At the point B in the straight line AB, let the two straight lines BC, BD upon the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles; then BD shall be in the same straight line with CB.



For, if BD be not in the same straight line with CB, let BE be in the same straight line with it; therefore, because the straight line AB makes with the straight line CBE, upon one side of it, the angles ABC, ABE, these angles are together equal* to two right angles; but the angles ABC, ABD are likewise together equal to two right angles; therefore the angles CBA, ABE are equal to the angles CBA, ABD: take away the common angle ABC, and the remaining angle ABE is

* 13. 1.

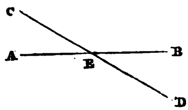
* 3 Ax. equal* to the remaining angle ABD, the less to the greater, which is impossible; therefore BE is not in the same straight line with BC. And, in like manner, it may be demonstrated, that no other can be in the same straight line with it but BD, which therefore is in the same straight line with CB. Wherefore, if at a point, &c. Q. E. D.

PROP. XV. THEOR.

If two straight lines cut one another, the vertical, or opposite, *angles shall be equal.*

Let the two straight lines AB, CD cut one another in the point E; the angle AEC shall be equal to the angle DEB, and CEB to AED.

Because the straight line AE makes with CD the angles CEA, AED, these
* 13. 1. angles are together equal* to two right angles. Again, because the straight line DE makes with AB the angles



* 13. 1. AED, DEB, these also are together equal* to two right angles; and the angles CEA, AED have been demonstrated to be together equal to two right angles; wherefore the angles CEA, AED are equal to the angles AED, DEB. Take away the common angle
* 3 Ax. AED, and the remaining angle CEA is equal* to the remaining angle DEB. In the same manner it can be demonstrated that the angles CEB, AED are equal. Therefore, if two straight lines, &c. Q. E. D.

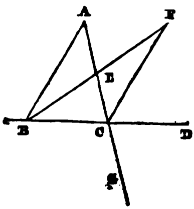
Cor. 1. From this it is manifest, that, if two straight lines cut one another, the angles which they make at the point where they cut, are together equal to four right angles.

Cor. 2. And consequently that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

PROP. XVI. THEOR.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Let ABC be a triangle, and let its side BC be produced to D, the exterior angle ACD shall be greater than either of the interior opposite angles CBA, BAC.



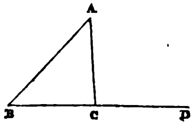
Bisect* AC in E, join BE and in BE produced make EF equal to BE, and join FC. * 10. 1.

Because AE is equal to EC, and BE to EF; AE, EB are equal to CE, EF, each to each; and the angle AEB is equal* to the angle CEF, because they are opposite vertical angles; therefore the base AB is equal* to the base CF, and the triangle AEB to the triangle CEF, and the remaining angles to the remaining angles, each to each, to which the equal sides are opposite; wherefore the angle BAE is equal to the angle ECF; but the angle ECD is greater than the angle ECF; therefore the angle ACD is greater than BAE. In like manner, if the side BC be bisected, and AC be produced to G, it may be demonstrated that the angle BCG, that is,* the angle ACD, is greater than the angle ABC. Therefore, if one side, &c. Q. E. D. * 15. 1. * 4. 1. * 15. 1.

PROP. XVII. THEOR.

Any two angles of a triangle are together less than two right angles.

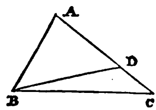
Let ABC be any triangle; any two of its angles together shall be less than two right angles.



Produce BC to D; and because ACD is the exterior angle of the triangle ABC, ACD is
* 16. 1. greater* than the interior and opposite angle ABC; to each of these add the angle ACB; therefore the angles ACD, ACB are greater than the angles
* 13. 1. ABC, ACB; but ACD, ACB are together equal* to two right angles; therefore the angles ABC, BCA are less than two right angles. In like manner, it may be demonstrated, that BAC, ACB, as also CAB, ABC, are less than two right angles. Therefore any two angles, &c. Q. E. D.

PROP. XVIII. THEOR.

The greater side of every triangle is opposite to the greater angle.



Let ABC be a triangle, of which the side AC is greater than the side AB; the angle ABC shall be greater than the angle BCA.

Because AC is greater than AB,
* 3. 1. make* AD equal to AB, and join BD; and because ADB is the exterior angle of the triangle BDC, it is
* 16. 1. greater* than the interior and opposite angle DCB;
* 5. 1. but ADB is equal* to ABD, because the side AB is equal to the side AD; therefore the angle ABD is likewise greater than the angle ACB; much more then is the angle ABC greater than ACB. Therefore the greater side, &c. Q. E. D.

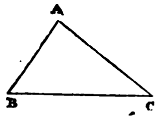
PROP. XIX. THEOR.

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

Let ABC be a triangle of which the angle ABC is

greater than the angle BCA; the side AC shall be greater than the side AB.

For, if it be not greater, AC must either be equal to AB, or less than it; it is not equal, because then the angle ABC would be equal* to the angle ACB; but it * 5. 1.
is not; therefore AC is not equal to AB: neither is it less; because then the angle ABC would be less* than the angle * 18. 1.
ACB; but it is not: therefore the side AC is not less than AB; and it has been shown that it is not equal to AB; therefore AC is greater than AB. Wherefore the greater angle, &c. Q. E. D.

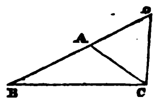


PROP. XX. THEOR.

Any two sides of a triangle are together greater than the third side.

Let ABC be a triangle: any two sides of it together shall be greater than the third side, viz. the sides BA, AC greater than the side BC; and AB, BC greater than AC; and BC, CA greater than AB.

In BA produced take AD equal to AC; and join DC.



Because AD is equal to AC, the angle ADC is equal* to ACD; but * 5. 1.
the angle BCD is greater than the angle ACD; therefore the angle BCD is greater than the angle ADC; and because the angle BCD of the triangle DCB is greater than its angle BDC, and that the greater* angle is subtended by the greater side; * 19. 1.
therefore the side DB is greater than the side BC; but BD is equal to BA and AC, because AC is equal to AD; therefore the sides BA, AC are greater than BC. In the same manner it may be demonstrated, that the

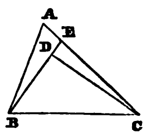
sides AB, BC are greater than CA, and BC, CA greater than AB. Therefore any two sides, &c. Q. E. D.

PROP. XXI. THEOR.

If from the ends of the side of a triangle, there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let ABC be a triangle, and from the points B, C, the ends of the side BC, let the two straight lines BD, CD be drawn to a point D within the triangle; BD and DC shall be less than the two sides BA, AC of the triangle, but shall contain an angle BDC greater than the angle BAC.

Produce BD to meet AC in E; and because two sides of a triangle are greater than the third side, the two sides BA, AE of the triangle ABE are greater than BE. To each of these unequals add EC; therefore the sides BA, AC are greater than BE, EC. Again, because the two sides CE, ED of the triangle CED are greater than CD, add DB to each of these unequals; therefore the sides CE, EB are greater than CD, DB; but it has been shown that BA, AC are greater than BE, EC; much more then are BA, AC greater than BD, DC.



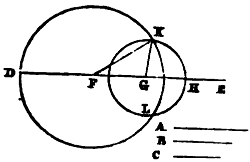
Again, because the exterior angle of a triangle is greater than the interior and opposite angle, the exterior angle BDC of the triangle CDE is greater than CED; for the same reason, the exterior angle CEB of the triangle ABE is greater than BAC; and it has been demonstrated that the angle BDC is greater than the angle CEB; much more then is the angle BDC greater than the angle BAC. Therefore, if from the *ends of, &c.* Q. E. D.

PROP. XXII. PROB.

*To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.** * 20. 1.

Let A, B, C, be the three given straight lines, of which any two whatever are greater than the third, viz. A and B greater than C; A and C greater than B; and B and C than A. It is required to make a triangle of which the sides shall be equal to A, B, C, each to each.

Take a straight line DE terminated at the point D, but unlimited towards E, and make * DF equal * 3. 1.
to A, FG equal to B, and GH equal to C; and from the centre F, at the distance FD, describe* the circle DKL; * 3 Post.
and from the centre G,



at the distance GH, describe* another circle HLK; * 3 Post.
and join KF, KG; the triangle KFG shall have its sides equal to the three straight lines, A, B, C.

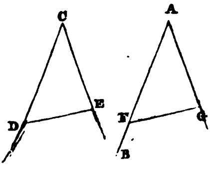
Because the point F is the centre of the circle DKL, FD is equal* to FK; but FD is equal to the straight * 15 Def.
line A; therefore FK is equal to A. Again, because G is the centre of the circle LKH, GH is equal* to GK; * 15 Def.
but GH is equal to C; therefore also GK is equal to C; and FG is equal to B; therefore the three straight lines KF, FG, GK, are respectively equal to the three A, B, C: and therefore the triangle KFG has its three sides KF, FG, GK, equal to the three given straight lines A, B, C. Which was to be done.

PROP. XXIII. PROB.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, and A the given

point in it, and DCE the given rectilineal angle; it is required to make an angle at the given point A in the given straight line AB, that shall be equal to the given rectilineal angle DCE.



In CD, CE, take any points D, E, and join DE; and
* 22. 1. make* the triangle AFG the sides of which shall be equal to the three straight lines CD, DE, CE, so that CD be equal to AF, CE to AG, and DE to FG; then shall the angle FAG be equal to the given angle C. Because FA, AG are equal to DC, CE, each to each, and the base FG to the base DE;
* 8. 1. therefore the angle FAG is equal* to the angle DCE. Wherefore, at the given point A in the given straight line AB, the angle FAG is made equal to the given rectilineal angle DCE. Which was to be done.

PROP. XXIV. THEOR.

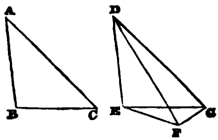
If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them, of the other; the base of that which has the greater angle shall be greater than the base of the other.

Let ABC, DEF be two triangles, which have the two sides AB, AC equal to the two DE, DF, each to each, viz. AB equal to DE, and AC to DF; but the angle BAC greater than the angle EDF; the base BC shall be greater than the base EF.

Of the two sides DE, DF, let DE be the side which is not greater than the other, and at the point D, in
* 23. 1. the straight line DE, make* the angle EDG equal to
* 3. 1. the angle BAC; and make DG equal* to AC or DF, and join EG, GF.

Because DE is equal to AB, and DG to AC, the

two sides ED, DG are equal to the two BA, AC, each to each, and the angle EDG is equal to the angle BAC; therefore the base EG is equal* to the base BC; and because DG is equal to DF, the angle DFG is equal* to the angle DGF; but the angle DGF is greater than the angle EGF; therefore the angle DFG is greater than the angle EGF; much more then is the angle EFG greater than the angle EGF; and because the angle EFG of the triangle EFG is greater than its angle EGF, and that the greater* angle is subtended by the greater side; therefore the side EG is greater than the side EF; but EG has been proved to be equal to BC; and therefore BC is greater than EF. Therefore, if two triangles, &c. Q. E. D.



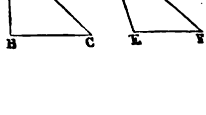
* 4. 1.
* 5. 1.
* 19. 1.

PROP. XXV. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other; the angle contained by the sides of that which has the greater base, shall be greater than the angle contained by the sides equal to them of the other.

Let ABC, DEF be two triangles which have the two sides AB, AC equal to the two sides DE, DF, each to each, viz. AB equal to DE, and AC to DF; but the base CB greater than the base EF; the angle BAC shall be greater than the angle EDF.

For, if it be not greater, it must either be equal to it, or less than it; but the angle BAC is not equal to the angle EDF, because then the base BC would be equal* to EF;



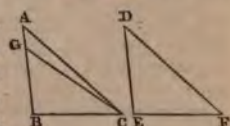
* 4. 1.

but it is not; therefore the angle BAC is not equal to the angle EDF; neither is it less; because then the
* 24. 1. base BC would be less* than the base EF; but it is not; therefore the angle BAC is not less than the angle EDF; and it was proved that it is not equal to it; therefore the angle BAC is greater than the angle EDF. Wherefore, if two triangles, &c. Q. E. D.

PROP. XXVI. THEOR.

If two triangles have two angles of the one equal to two angles of the other, each to each; and one side equal to one side, viz. either the sides adjacent to the equal angles, or the sides opposite to the equal angles in each; then shall the other sides be equal, each to each; and the third angle of the one equal to the third angle of the other.

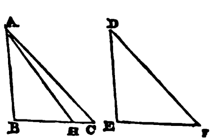
Let ABC, DEF be two triangles which have the angles ABC, BCA equal to the angles DEF, EFD, each to each, viz. ABC to DEF, and BCA to EFD; also one side equal to one side; and first let those sides be equal which are adjacent to the angles that are equal in the two triangles, viz. BC to EF; the other sides shall be equal, each to each, viz. AB to DE, and AC to DF; and the third angle BAC to the third angle EDF.



For, if AB be not equal to DE, one of them must be greater than the other. Let AB be the greater, and make BG equal to ED, and join GC; therefore, because BG is equal to ED, and BC to EF, the two sides GB, BC are equal to the two DE, EF, each to each; and the angle GBC is equal to the angle DEF; therefore the triangle GBC is equal to the triangle DEF, and the other angles to the other angles, each to each, to which the equal sides are opposite; therefore the

angle GCB is equal to the angle* DFE; but DFE is, by the hypothesis, equal to the angle ACB; wherefore also the angle BCG is equal to the angle BCA, the less to the greater, which is impossible; therefore AB is not unequal to DE, that is, it is equal to it; and BC is equal to EF; therefore the two sides AB, BC are equal to the two DE, EF, each to each; and the angle ABC is equal to the angle DEF; therefore the base AC is equal* to the base DF, and the third angle BAC to the third angle EDF. * 1 Ax. * 4. 1.

Next, let the sides which are opposite to the equal angles in each triangle be equal to one another, viz. AB to DE; likewise in this case, the other sides shall be equal, viz. AC to DF, and BC to EF; and also the third angle BAC to the third angle EDF.



For, if BC be not equal to EF, let BC be the greater, and make BH equal* to EF, and join AH; then, because BH is equal to EF, and AB to DE, the two sides AB, BH are equal to the two DE, EF, each to each; and they contain equal angles; therefore the triangle ABH is equal* to the triangle DEF, and the other angles to the other angles, each to each, to which the equal sides are opposite; therefore the angle BHA is equal to the angle EFD; but EFD is equal to the angle BCA; therefore the angle BHA is also equal* to the angle BCA, that is, the exterior angle BHA of the triangle AHC is equal to its interior and opposite angle HCA; which is impossible:* wherefore BC is not unequal to EF, that is, it is equal to it; and AB is equal to DE; therefore the two sides AB, BC are equal to the two DE, EF, each to each; and they contain equal angles; wherefore the base AC is equal* to the base * 3. 1. * 4. 1. * 1 Ax. * 16. 1. * 4. 1.

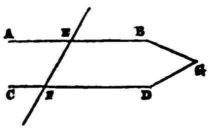
DF, and the third angle BAC to the third angle EDF. Therefore, if two triangles, &c. Q. E. D.

PROP. XXVII. THEOR.

If a straight line, falling upon two other straight lines, make the alternate angles equal to one another, these two straight lines shall be parallel.

Let the straight line EF, which falls upon the two straight lines AB, CD make the alternate angles AEF, EFD equal to one another; AB shall be parallel to CD.

For, if AB be not parallel to CD, AB and CD being produced will meet either towards B, D, or towards A, C; let them be produced and meet towards B, D in the point G; therefore GEF is a triangle, and its ex-
* 16. 1. terior angle AEF is greater* than the interior and opposite angle EFG; but this is impossible, because the angle AEF is equal to the angle EFG, by hypothesis; therefore AB and CD being produced do not meet towards B, D. In like manner it may be demonstrated that they do not meet towards A, C; but those straight lines which do not meet, though produced either way
* 35 Def. continually, are parallel* to one another: therefore AB is parallel to CD. Wherefore, if a straight line, &c. Q. E. D.

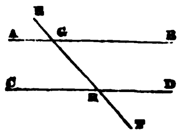


PROP. XXVIII. THEOR.

If a straight line, falling upon two other straight lines, makes the exterior angle equal to the interior and opposite upon the same side of the line; or make the interior angles upon the same side together equal to two right angles; the two straight lines shall be parallel to one another.

Let the straight line EF, which falls upon the two straight lines AB, CD, make the exterior angle EGB

equal to the interior and opposite angle GHD upon the same side; or make on the same side the interior angles BGH, GHD together equal to two right angles; AB shall be parallel to CD.



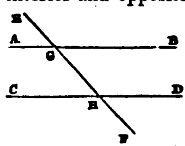
Because the angle EGB is equal* to the angle GHD, and the angle EGB equal* to the angle AGH, therefore the angle AGH is equal to the angle GHD; and they are alternate angles; therefore AB is parallel* to CD. Again, because the angles BGH, GHD are together equal* to two right angles, and AGH, BGH, are also together equal* to two right angles; therefore the angles AGH, BGH are equal* to the angles BGH, GHD: take away the common angle BGH; and the remaining angle AGH is therefore equal* to the remaining angle GHD; and they are alternate angles: therefore AB is parallel to CD. Wherefore, if a straight line, &c. Q. E. D.

* Hyp. * 15. 1. * 27. 1. * Hyp. * 13. 1. * 1 Ax. * 3 Ax.

PROP. XXIX. THEOR.

If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and opposite upon the same side; and likewise the two interior angles upon the same side together equal to two right angles.

Let the straight line EF fall upon the parallel straight lines AB, CD; the alternate angles AGH, GHD shall be equal to one another; and the exterior angle EGB shall be equal to the interior and opposite angle GHD, upon the same side; and the two interior angles BGH, GHD upon the same side, shall be together equal to two right angles.



For, if AGH be not equal to

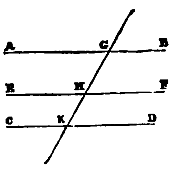
GHD, one of them must be greater than the other: let AGH be the greater; and because the angle AGH is greater than the angle GHD, add to each of these unequals the angle BGH; therefore the angles AGH,
* 4 Ax. BGH are greater* than the angles BGH, GHD; but
* 13. 1. the angles AGH, BGH are together equal* to two right angles; therefore the angles BGH, GHD are less than two right angles; but those straight lines which, with another straight line falling upon them, make the interior angles on the same side less than two right
* 12 Ax. angles meet,* if far enough produced; therefore the straight lines AB, CD, if produced far enough, will meet; but they never meet, since they are parallel by the hypothesis; therefore the angle AGH is not unequal to the angle GHD, that is, it is equal to it: but
* 15. 1. the angle AGH is equal* to the angle EGB; therefore EGB is also equal to GHD: add to each of these equals the angle BGH; then will the angles EGB, BGH be
* 2 Ax. equal* to the angles BGH, GHD; but EGB, BGH are
* 13. 1. equal* to two right angles; therefore BGH, GHD are
* 1 Ax. also equal* to two right angles. Wherefore, if a straight, &c. Q. E. D.

PROP. XXX. THEOR.

Straight lines which are parallel to the same straight line are parallel to each other.

Let AB, CD be each of them parallel to EF; AB shall be parallel to CD.

Let the straight line GHK cut AB, EF, CD; and because GHK cuts the parallel straight lines AB, EF, the angle
* 29. 1. AGH is equal* to the alternate angle GHF. Again, because the straight line GK cuts the parallel straight lines EF, CD, the angle
* 29. 1. GHF is equal* to the angle GKD;

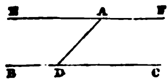


and it was proved that the angle AGK is equal to the angle GHF; therefore AGK is also equal to GKD; and they are alternate angles; therefore AB is parallel* to CD. Wherefore straight lines, &c. Q. E. D. * 27. 1.

PROP. XXXI. PROB.

To draw a straight line through a given point parallel to a given straight line.

Let A be the given point, and BC the given straight line; it is required to draw a straight line through the point A, parallel to the straight line BC.



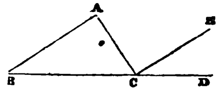
In BC take any point D, and join AD; and at the point A, in the straight line AD make* the angle DAE equal to the angle ADC; and produce the straight line EA to F. * 23. 1.

Because the straight line AD meets the two straight lines BC, EF, and makes the alternate angles EAD, ADC equal to one another, EF is parallel* to BC. Therefore the straight line EAF is drawn through the given point A, parallel to the given straight line BC. Which was to be done. * 27. 1.

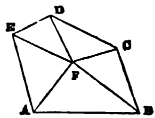
PROP. XXXII. THEOR.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles.

Let ABC be a triangle, and let one of its sides BC be produced to D; the exterior angle ACD shall be equal to the two interior and opposite angles CAB, ABC: and the three interior angles of the triangle, viz. ABC, BCA, CAB, shall be together equal to two right angles.



Through the point C draw
* 31. 1. CE parallel* to the straight line BA; and because CE is parallel to BA, and AC meets them, the alternate an-
* 29. 1. gles BAC, ACE are equal.* Again, because CE is parallel to AB, and BD falls upon them, the exterior angle
* 29. 1. ECD is equal* to the interior and opposite angle ABC; but the angle ACE was proved to be equal to the angle
* 2 Ax. BAC; therefore the whole exterior angle ACD is equal* to the two interior and opposite angles CAB, ABC; to each of these equals add the angle ACB, and the angles
* 2 Ax. ACD, ACB are equal* to the three angles CBA, BAC,
* 13. 1. ACB; but the angles ACD, ACB are equal* to two right angles; therefore the angles CBA, BAC, ACB,
* 1 Ax. are also equal* to two right angles. Wherefore, if a side of a triangle, &c. Q. E. D.

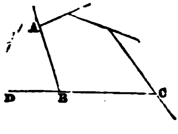


Cor. 1. All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

For any rectilineal figure ABCDE can be divided into as many triangles as the figure has sides, by drawing straight lines from a point F within the figure to each of its angles. And, by the preceding proposition, all the angles of these triangles are equal to twice as many right angles as there are triangles, that is, as there are sides of the figure; and the same angles are equal to the angles of the figure, together with the angles at the point F, which is the common vertex of
* 2 Cor. 15. 1. the triangles; that is* together with four right angles. Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

Cor. 2. All the exterior angles of any rectilineal figure are together equal to four right angles.

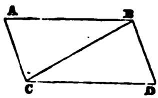
Because every interior angle ABC, with its adjacent exterior angle ABD, is equal* to two right angles; * 13. 1.
therefore all the interior, together with all the exterior angles of the figure, are equal to twice as many right angles as there are sides of the figure; that is, by the foregoing corollary, they are equal to all the interior angles of the figure, together with four right angles; therefore all the exterior angles are equal to four right angles.



PROP. XXXIII. THEOR.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.

Let AB, CD be equal and parallel straight lines, and joined towards the same parts by the straight lines AC, BD; AC, BD shall be equal and parallel.



Join BC; and because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal;* and because AB is equal to CD, and BC com- * 29. 1.
mon to the two triangles ABC, DCB, the two sides AB, BC are equal to the two DC, CB; and the angle ABC was proved to be equal to the angle BCD; therefore the base AC is equal* to the base BD, and * 4. 1.
the triangle ABC to the triangle BCD, and the other angles to the other angles,* each to each, to which the * 4. 1.
equal sides are opposite; therefore the angle ACB is equal to the angle CBD; and because the straight line BC meets the two straight lines AC, BD, and makes

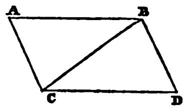
the alternate angles ACB, CBD equal to one another,
* 27. 1. AC is parallel* to BD; and it was proved to be equal to it. Therefore, straight lines, &c. Q. E. D.

PROP. XXXIV. THEOR.

The opposite sides and angles of parallelograms are equal to one another, and the diameter bisects them, that is, divides them into two equal parts.

N. B. A parallelogram is a four-sided figure, of which the opposite sides are parallel: and the diameter is the straight line joining two of its opposite angles.

Let ACDB be a parallelogram, of which BC is a diameter: the opposite sides and angles of the figure shall be equal to one another; and the diameter BC shall bisect it.



Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are
* 29. 1. equal* to one another; and because AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD are
* 29. 1. equal* to one another; wherefore the two triangles ABC, CBD have two angles ABC, BCA in the one, equal to two angles BCD, CBD in the other, each to each, and one side BC, which is adjacent to their equal angles, common to the two triangles; therefore their other sides are equal, each to each, and the third angle
* 26. 1. of the one to the third angle of the other,* viz. the side AB to the side CD, and AC to BD, and the angle BAC equal to the angle BDC: and because the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB, the whole angle ABD is equal to the whole angle ACD: and the angle BAC has been proved to be equal to the angle BDC; therefore the opposite sides and angles of parallelograms are equal to one another; also, their diameter bisects them; for AB being equal to CD, and BC common, the two AB,

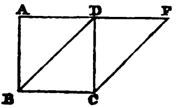
BC are equal to the two DC, CB, each to each; and the angle ABC has been proved equal to the angle BCD; therefore the triangle ABC is equal* to the triangle BCD, and the diameter BC divides the parallelogram ACDB into two equal parts. Q. E. D. * 4. 1.

PROP. XXXV. THEOR.

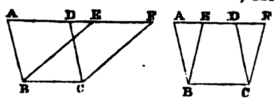
Parallelograms upon the same base, and between the same parallels, are equal to one another.

Let the parallelograms ABCD, DBCF be upon the same base BC, and between the same parallels AF, BC; the parallelogram ABCD shall be equal to the parallelogram DBCF. See the 2d and 3d figures.

If the sides AD, DF of the parallelograms ABCD, DBCF opposite to the base BC be terminated in the same point D; it is plain that each of the parallelograms is double* of the triangle BDC; and they are therefore equal* to one another. * 34. 1. * 6 Ax.



But, if the sides AD, EF, opposite to the base BC of the parallelograms ABCD, EBCF, be not terminated in the same point; then, because ABCD is a parallelogram, AD is equal* to BC; for the same reason EF is equal to BC; wherefore AD is equal* to EF; and DE is common; therefore the whole, or the remainder, AE is equal* to the whole, or the remainder DF; AB is also equal to DC; and therefore the two EA, AB are equal to the two FD, DC, each to each; and the exterior angle FDC is equal* to the interior EAB, therefore the base EB is equal to the base FC, and the triangle EAB equal* to the triangle FDC; take the triangle FDC from the trapezium * 34. 1. * 1 Ax. * 2 or 3 Ax. * 29. 1. * 4. 1.

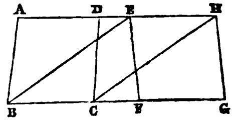


ABCF, and from the same trapezium take the triangle EAB, which has been proved equal to the triangle
* 3 Ax. FDC; and the remainders are equal,* that is, the parallelogram ABCD is equal to the parallelogram EBCF. Therefore parallelograms upon the same base, &c. Q. E. D.

PROP. XXXVI. THEOR.

Parallelograms upon equal bases, and between the same parallels, are equal to one another.

Let ABCD, EFGH be parallelograms upon equal bases BC, FG, and between the same parallels AH, BG; the parallelogram ABCD shall be equal to EFGH.



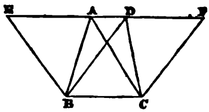
* Hyp. Join BE, CH; and because BC is equal* to FG, and
* 34. 1. FG to* EH, BC is equal to EH; and they are parallels, and joined towards the same parts by the straight lines BE, CH: but straight lines which join the extremities of equal and parallel straight lines towards
* 33. 1. the same parts, are themselves equal and parallel;* therefore EB, CH are both equal and parallel, and
* 35. 1. therefore EBCH is a parallelogram; and it is equal* to ABCD, because they are upon the same base BC, and between the same parallels BC, AH: for the same reason, the parallelogram EFGH is equal to the same EBCH: therefore the parallelogram ABCD is equal to EFGH. Wherefore parallelograms, &c. Q. E. D.

PROP. XXXVII. THEOR.

Triangles upon the same base, and between the same parallels, are equal to one another.

Let the triangles ABC, DBC be upon the same base

BC, and between the same parallels AD, BC: the triangle ABC shall be equal to the triangle DBC.



Produce AD both ways, and through B draw* BE parallel to CA; and through C draw CF parallel to BD: therefore each of the figures EBCA, DBCF is a parallelogram; and EBCA is equal* to DBCF, because they are upon the same base BC, and between the same parallels BC, EF; and the triangle ABC is the half of the parallelogram EBCA, because the diameter AB bisects* it; and the triangle DBC is the half of the parallelogram DBCF, because the diameter DC bisects it: but the halves of equal things are equal;* therefore the triangle ABC is equal to the triangle DBC. Wherefore triangles, &c. Q. E. D.

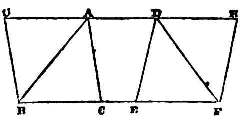
* 31. 1.
* 35. 1.
* 34. 1.
* 7 Ax.

PROP. XXXVIII. THEOR.

Triangles upon equal bases, and between the same parallels, are equal to one another.

Let the triangles ABC, DEF be upon equal bases BC, EF, and between the same parallels BF, AD: the triangle ABC shall be equal to the triangle DEF.

Produce AD both ways, and through B draw BG parallel* to CA, and through F draw FH parallel to ED: then each of the figures GBCA, DEFH is a parallelogram; and they are equal to* one another, because they are upon equal bases BC, EF, and between the same parallels BF, GH; and the triangle ABC is the half* of the parallelogram GBCA, because the diameter AB bisects it; and the triangle DEF is the half* of the parallelogram DEFH, be-



* 31. 1.
* 36. 1.
* 34. 1.
* 34. 1

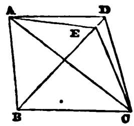
cause the diameter DF bisects it: but the halves of
* 7 Ax. equal things are equal;* therefore the triangle ABC is equal to the triangle DEF. Wherefore triangles, &c. Q. E. D.

PROP. XXXIX. THEOR.

Equal triangles, upon the same base and upon the same side of it, are between the same parallels.

Let the equal triangles ABC, DBC be upon the same base BC, and upon the same side of it; they shall be between the same parallels.

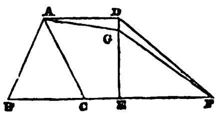
Join AD; AD shall be parallel to BC; for, if it is
* 31. 1. not, through the point A draw* AE parallel to BC, and
* 37. 1. join EC; then the triangle ABC is equal* to the triangle EBC, because they are upon the same base BC, and between the same parallels BC, AE: but the triangle ABC is equal to the triangle DBC; therefore the triangle BDC is also equal to the triangle EBC, the greater to the less, which is impossible: therefore AE is not parallel to BC. In the same manner, it can be demonstrated that no other line but AD is parallel to BC; therefore AD is parallel to BC. Wherefore equal triangles upon, &c. Q. E. D.



PROP. XL. THEOR.

Equal triangles, upon equal bases in the same straight line, and towards the same parts, are between the same parallels.

Let the equal triangles ABC, DEF be upon equal bases BC, EF, in the same straight line BF, and towards the same parts; they shall be between the same parallels.

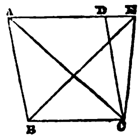


Join AD: AD shall be parallel to BC; for, if it is not, through A draw* AG parallel to BF, and join * 31. 1.
GF. The triangle ABC is equal* to the triangle GEF, * 38. 1.
because they are upon equal bases BC, EF, and between the same parallels BF, AG: but the triangle ABC is equal to the triangle DEF; therefore the triangle DEF is also equal to the triangle GEF, the greater to the less, which is impossible: therefore AG is not parallel to BF. And in the same manner it can be demonstrated that there is no other parallel to it but AD; therefore AD is parallel to BF. Wherefore equal triangles, &c. Q. E. D.

PROP. XLI. THEOR.

If a parallelogram and a triangle be upon the same base, and between the same parallels; the parallelogram shall be double of the triangle.

Let the parallelogram ABCD and the triangle EBC be upon the same base BC, and between the same parallels BC, AE; the parallelogram ABCD shall be double of the triangle EBC.



Join AC; then the triangle ABC is equal* to the triangle EBC, because they are upon the same base BC, and between the same parallels BC, AE. But the parallelogram ABCD is double* of the triangle ABC, because the diameter AC divides it into two equal parts; wherefore ABCD is also double of the triangle EBC. Therefore, if a parallelogram, &c. Q. E. D.

* 37. 1.
* 34. 1.

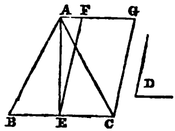
PROP. XLII. PROB.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given

rectilineal angle. It is required to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.

* 10. 1. Bisect* BC in E, join AE, and at the point E in the
* 23. 1. straight line EC make* the angle CEF equal to D;
* 31. 1. and through A draw* AG parallel to EC, and through
* 31. 1. C draw* CG parallel to EF: therefore FECG is a parallelo-
* Def. 34. 1. gram.* And because BE is equal to EC, the triangle ABE is
* 38. 1. equal* to the triangle AEC, since they are upon equal bases

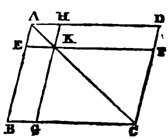


BE, EC, and between the same parallels BC, AG; therefore the triangle ABC is double of the triangle
* 41. 1. AEC. But the parallelogram FECG is likewise double* of the triangle AEC, because they are upon the same base EC, and between the same parallels EC, AG:
* 6 Ax. therefore the parallelogram FECG is equal* to the triangle ABC, and it has one of its angles CEF equal to the given angle D: wherefore a parallelogram FECG has been described equal to the given triangle ABC, having one of its angles CEF equal to the given angle D. Which was to be done.

PROP. XLIII. THEOR.

The complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.

Let ABCD be a parallelogram, of which the diameter is AC, and EH, GF the parallelograms about AC, *that is, through which* AC *passes,* and BK, KD the other parallelograms which make up the whole figure ABCD, which are therefore called



the complements: the complement BK shall be equal to the complement KD.

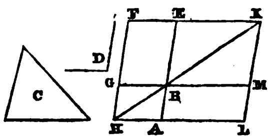
Because ABCD is a parallelogram, and AC its diameter, the triangle ABC is equal* to the triangle ADC. * 34. 1.
Again, because EKHA is a parallelogram, the diameter of which is AK, the triangle AEK is equal* to the triangle AHK: for a like reason, the triangle KGC is equal to the triangle KFC. But the triangle AEK was proved equal to the triangle AHK, and the triangle KGC to KFC; therefore the triangle AEK together with the triangle KGC is equal to the triangle AHK together with the triangle KFC: and the whole triangle ABC was proved equal to the whole ADC; therefore the remaining complement BK is equal* to the remaining complement KD. Wherefore the complements, &c. Q. E. D. * 34. 1. * 3 Ax.

PROP. XLIV. PROB.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, and C the given triangle, and D the given rectilineal angle. It is required to apply to the straight line AB a parallelogram equal to the triangle C, and having an angle equal to D.

Make* the parallelogram BEFG equal to the triangle C, and having the angle EBG equal to the angle D, so that BE be in the same straight line with AB, that is, in AB produced, and produce FG to H; and through A draw* AH parallel to BG or EF, and join HB. Then, because the straight line HF falls upon the parallels AH, EF, * 42. 1. * 31. 1.



* 29 1. the angles AHF, HFE, are together equal* to two right angles; therefore the angles BHF, HFE are less than two right angles: but straight lines, which, with another straight line, make the interior angles upon
* 12 Ax. the same side less than two right angles, will meet* if produced far enough: therefore HB, FE will meet, if produced; let them meet in K, and through K draw KL parallel to EA or FH, and produce HA, GB to the points L, M: then HLKF is a parallelogram, of which the diameter is HK, and AG, ME are the parallelograms about HK; and LB, BF are the comple-
* 43. 1. ments; therefore LB is equal* to BF: but BF is equal to the triangle C; wherefore LB is equal to the tri-
* 15. 1. angle C; and because the angle GBE is equal* to the angle ABM, and likewise to the angle D; the angle ABM is equal to the angle D. Therefore, to the straight line AB, the parallelogram LB is applied equal to the triangle C, having the angle ABM equal to the angle D. Which was to be done.

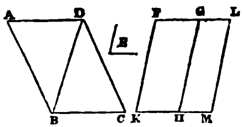
PROP. XLV. PROB.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilineal figure, and E the given rectilineal angle. It is required to describe a parallelogram that shall be equal to ABCD, and have an angle equal to E.

* 42. 1. Join DB; and describe* the parallelogram FH equal to the triangle ADB, and having the angle HKF equal to the angle E; and to the straight line GH
* 44. 1. apply* the parallelogram GM equal to the triangle DBC, having the angle GHM equal to the angle E: then FKLM shall be the parallelogram required.

Because the angle E is equal to each of the angles FKH, GHM, the angle FKH is equal to GHM; add



to each of these equals the angle KHG; therefore the angles FKH, KHG are equal to the angles KHG, GHM; but FKH, KHG are equal* to two right angles: therefore KHG, GHM are also equal to two right angles; and because at the point H, in the straight line GH, the two straight lines KH, HM, upon the opposite sides of it, make the adjacent angles equal to two right angles, KH is in the same straight line* with HM; and because the straight line HG meets the parallels KM, FG, the alternate angles MHG, HGF are equal:* add to each of these equals the angle HGL; therefore the angles MHG, HGL, are equal to the angles HGF, HGL: but the angles MHG, HGL are equal* to two right angles; wherefore the angles HGF, HGL are also equal to two right angles, and FG is therefore in the same straight line* with GL: and because KF is parallel to HG, and HG to ML, KF is parallel* to ML: and KM, FL are parallels; wherefore KFLM is a parallelogram; and because the triangle ABD is equal to the parallelogram HF, and the triangle DBC to the parallelogram GM; the whole rectilineal figure ABCD is equal to the whole parallelogram KFLM; therefore the parallelogram KFLM has been described equal to the given rectilineal figure ABCD, having the angle FKM equal to the given angle E. Which was to be done.

* 29. 1.
* 14. 1.
* 29. 1.
* 29. 1.
* 14. 1.
* 30. 1.

COR. From this it is manifest how to a given straight line to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure, viz. by applying* to the given straight line a parallelogram equal to the first triangle ABD, and having an angle equal to the given angle.

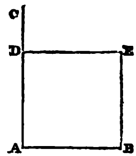
* 44. 1.

PROP. XLVI. PROB.

To describe a square upon a given straight line.

Let AB be the given straight line; it is required to describe a square upon AB.

* 11. 1. From the point A draw* AC at right angles to AB;
* 3. 1. and make* AD equal to AB, and through the point D
* 31. 1. draw DE parallel* to AB, and through B draw BE parallel to AD; then ABCD shall be the square required. Since ADEB is a parallelogram, therefore
* 34. 1. AB is equal* to DE, and AD to BE: but BA is equal to AD; therefore the four straight lines BA, AD, DE, EB are equal to one another, and the parallelogram ADEB is equilateral. Likewise all its angles are right angles; for, since the straight line AD meets the parallels AB, DE, the angles BAD, ADE



* 29. 1. are equal* to two right angles; but BAD is a right angle; therefore also ADE is a right angle; but the
* 34. 1. opposite angles of parallelograms are equal;* therefore each of the opposite angles ABE, BED is a right angle; therefore the figure ADEB is rectangular, and it has been demonstrated that it is equilateral; it is therefore a square, and it is described upon the given straight line AB. Which was to be done.

Cor. Hence every parallelogram that has one right angle has all its angles right angles.

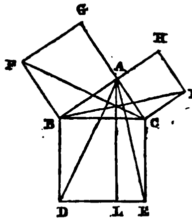
PROP. XLVII. THEOR.

In any right-angled triangle, the square which is described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle.

Let ABC be a right-angled triangle having the right angle BAC; the square described upon the side

BC shall be equal to the squares described upon BA, AC.

On BC describe* the square BDEC, and on BA, * 46. 1.
AC the squares GB, HC; and through A draw* AL * 31. 1.
parallel to BD, or CE, and join AD, FC; then, be-
cause each of the angles BAC, BAG is a right angle*, * 30 Def.
the two straight lines AC, AG upon the opposite sides
of AB, make with it at the point A the adjacent angles equal to two right angles; therefore CA is in the same
straight line* with AG; for * 14. 1.
the same reason, AB and AH are in the same straight line; and because the angle DBC is equal to the angle FBA, each of them being a right



angle, add to each of these equals the angle ABC,
therefore the whole angle DBA is equal* to the whole * 2 Ax.
FBC; and because the two sides AB, BD are equal
to the two FB, BC, each to each, and the angle ABD
equal to the angle FBC; therefore the triangle ABD
is equal* to the triangle FBC: but the parallelogram * 4. 1.
BL is double* of the triangle ABD, because they are * 41. 1.
upon the same base BD, and between the same parallels BD, AL; and the square GB is double of the triangle FBC, because these also are upon the same base FB, and between the same parallels FB, GC. But the
doubles of equals are equal* to one another: therefore * 6 Ax.
the parallelogram BL is equal to the square GB: and, in like manner, by joining AE, BK; it can be demonstrated that the parallelogram CL is equal to the square HC: therefore the whole square BDEC is equal to the two squares GB, HC; and the square BDEC is described upon the straight line BC, and the squares GB, HC upon BA, AC: therefore the square

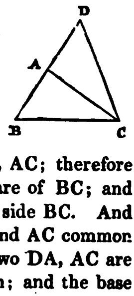
upon the side BC is equal to the squares upon the sides BA, AC. Therefore, in any right-angled triangle, &c. Q. E. D.

PROP. XLVIII. THEOR.

If the square described upon one of the sides of a triangle, be equal to the squares described upon the other two sides of it, the angle contained by these two sides is a right angle.

Let the square described upon BC, one of the sides of the triangle ABC, be equal to the squares upon the other sides BA, AC: the angle BAC shall be a right angle.

* 11. 1. From the point A draw* AD at right angles to AC, and make AD equal to BA, and join DC: then, because DA is equal to AB, the square of DA is equal to the square of AB: to each of these equals add the square of AC; therefore the squares of DA, AC, are equal to the squares of BA, AC: but the



* 47. 1 square of DC is equal* to the squares of DA, AC, because DAC is a right angle; and the square of BC, by hypothesis, is equal to the squares of BA, AC; therefore the square of DC is equal to the square of BC; and therefore the side DC is equal to the side BC. And because the side DA is equal to AB, and AC common to the two triangles DAC, BAC, the two DA, AC are equal to the two BA, AC, each to each; and the base DC has been proved equal to the base BC; therefore

* 8. 1. the angle DAC is equal* to the angle BAC: but DAC is a right angle; therefore also BAC is a right angle. Therefore, if the square, &c. Q. E. D.

THE

ELEMENTS OF EUCLID.

BOOK II.

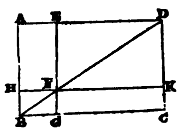
DEFINITIONS.

I.

Every right-angled parallelogram is called a rectangle, *and is said to be* contained *by any two of the straight lines which contain one of the right angles.*

II.

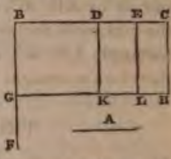
In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a *gnomon*. 'Thus the parallelogram 'HG, together with the complements AF, FC, is the gnomon, 'which is more briefly expressed 'by AGK, or EHC, the letters at the opposite angles 'of the parallelograms, which make the gnomon.'



PROP. I. THEOR.

If there be two straight lines, one of which is divided into any number of parts; the rectangle contained by the two straight lines, is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Let A and BC be two straight lines; and let BC be divided into any parts in the points D, E; the rectangle contained by the straight lines A, BC shall be equal to the rectangle contained by A, BD, together with that contained by A, DE, and that contained by A, EC.



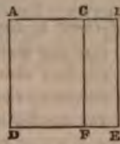
* 11. 1. From the point B draw* BF at right angles to BC, and make BG
* 3. 1. equal* to A; and through G
* 31. 1. draw* GH parallel to BC; and through D, E, C,
* 31. 1. draw* DK, EL, CH parallel to BG; then the rectangle BH is equal to the rectangles BK, DL, and EH; but BH is contained by A, BC, for it is contained by GB, BC, and GB is equal to A; and BK is contained by A, BD, for it is contained by GB, BD, of which GB is equal to A; and DL is contained by A, DE, because
* 34. 1. DK, that is* BG is equal to A; and in like manner the rectangle EH is contained by A, EC: therefore the rectangle contained by A, BC is equal to the rectangles contained by A, BD, by A, DE, and by A, EC. Wherefore, if there be two straight lines, &c. Q. E. D.

PROP. II. THEOR.

If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts, are together equal to the square of the whole line.

Let the straight line AB be divided into any two

parts in the point C; the rectangle contained by AB, BC, together with the rectangle[a] AB, AC, shall be equal to the square of AB.

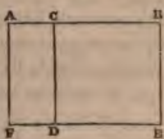


Upon AB describe* the square ADEB, and through C draw* CF, parallel to AD or BE; then AE is equal to the rectangles AF, CE; but AE is the square of AB; and AF is the rectangle contained by BA, AC; for it is contained by DA, AC, of which AD is equal to AB; and CE is contained by AB, BC, for BE is equal to AB; therefore the rectangle contained by AB, AC, together with the rectangle AB, BC, is equal to the square of AB. If therefore a straight line, &c. Q. E. D. * 46. 1. * 31. 1.

PROP. III. THEOR.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the square of the aforesaid part.

Let the straight line AB be divided into any two parts in the point C; the rectangle AB, BC shall be equal to the rectangle AC, CB, together with the square of BC.



Upon BC describe* the square CDEB, and produce ED to F, and through A draw* AF parallel to CD or BE; then the rectangle AE is equal to the rectangles AD, CE; but AE is the rectangle contained by AB, BC, for it is contained by AB, BE, of which BE is equal to BC; and AD is contained by AC, CB, * 46. 1. * 31. 1.

[a] To avoid repeating the word *contained* too frequently, the rectangle contained by two straight lines AB, AC is sometimes simply called *the rectangle AB, AC.*

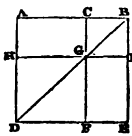
for CD is equal to CB; and BD is the square of BC; therefore the rectangle AB, BC is equal to the rectangle AC, CB, together with the square of BC. If therefore a straight, &c. Q. E. D.

PROP. IV. THEOR.

If a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

Let the straight line AB be divided into any two parts in C; the square of AB is equal to the squares of AC, CB, and to twice the rectangle contained by AC, CB.

* 46. 1. Upon AB describe* the square ADEB, and join BD,
* 31. 1. and through C draw* CGF parallel to AD or BE, and through G draw HK parallel to AB, or DE. Then, because CF is parallel to AD, and BD falls upon them,
* 29. 1. the exterior angle BGC is equal* to the interior and
* 5. 1. opposite angle ADB; but ADB is equal* to the angle ABD, because BA is equal to AD, being sides of a square; wherefore the angle CGB is equal to the angle CBG; and therefore the side CB is
* 6. 1. equal* to the side CG: but CB is
* 34. 1. equal* also to GK, and CG to BK; therefore the figure CGKB is equilateral; it is likewise rectangular; for since KBC is a right angle, by construction; therefore all the angles of the parallelogram



* Cor. 46 1. CGKB are right angles;* therefore CGKB is rectangular: but it is also equilateral, as was demonstrated; therefore it is a square, and it is upon the side CB; for the same reason HF is also a square, and it is upon the
* 34. 1. side HG, which is equal* to AC: therefore HF, CK are

the squares of AC, CB; and because the complement AG is equal* to the complement GE, and that AG is the rectangle contained by AC, CB, for GC is equal to CB; therefore GE is also equal to the rectangle AC, CB: therefore AG, GE are equal to twice the rectangle AC, CB: and HF, CK are the squares of AC, CB; wherefore the four figures HF, CK, AG, GE are equal to the squares of AC, CB, and to twice the rectangle AC, CB: but HF, CK, AG, GE make up the whole figure ADEB, which is the square of AB: therefore the square of AB is equal to the squares of AC, CB, and twice the rectangle AC, CB. Wherefore if a straight line, &c. Q. E. D. * 43. 1.

COR. From the demonstration, it is manifest that parallelograms about the diameter of a square are likewise squares.

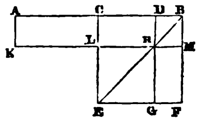
PROP. V. THEOR.

If a straight line be divided into two equal parts, and also into two unequal parts; the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half the line.

Let the straight line AB be divided into two equal parts in the point C, and into two unequal parts at the point D; the rectangle AD, DB, together with the square of CD, shall be equal to the square of CB.

Upon CB describe* the square CEFB, join BE, and * 46. 1.
through D draw* DHG parallel to CE or BF; and * 31. 1.
through H draw KLM parallel to CB or EF; and also through A draw AK parallel to CL or BM. And because the complement CH is equal* to the comple- * 43. 1.
ment HF, to each of these equals add DM; therefore the whole CM is equal to the whole DF; but CM is

* 36. 1. equal* to AL, because AC is equal to CB; therefore also AL is equal to DF: to each of these equals add CH, and the whole AH is equal to DF and CH: but AH is the rectangle contained by AD, DB,



* Cor. 4. 2. for DH is equal* to DB; and DF, together with CH, is the gnomon CMG; therefore the gnomon CMG is equal to the rectangle AD, DB: to

* Cor. 4. 2. each of these equals add LG, which is equal* to the square of CD; therefore the gnomon CMG, together with LG, is equal to the rectangle AD, DB, together with the square of CD: but the gnomon CMG and LG make up the whole figure CEFB, which is the square of CB; therefore the rectangle AD, DB, together with the square of CD, is equal to the square of CB. Wherefore, if a straight line, &c. Q. E. D.

From this proposition it is manifest, that the difference of the squares of two unequal lines AC, CD, is equal to the rectangle contained by their sum and difference.

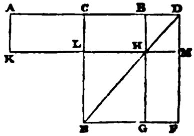
PROP. VI. THEOR.

If a straight line be bisected, and produced to any point, the rectangle contained by the whole line thus produced, and the part of it produced, together with the square of half the line bisected, is equal to the square of the straight line which is made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to the point D: the rectangle AD, DB, together with the square of CB, shall be equal to the square of CD.

* 46. 1. Upon CD describe* the square CEFD, join DE,
* 31. 1. and through B draw* BHG parallel to CE, or DF:

through H draw KLM parallel to AD or EF, and through A draw AK parallel to CL or DM. Then, since AC is equal to CB, the rectangle AL is equal* to CH; * 36. 1.
but CH is equal* to HF; therefore AL is also equal to HF: * 43. 1.
to each of these equals add CM; therefore the whole AM is



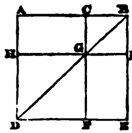
equal to the gnomon CMG: but AM is the rectangle contained by AD, DB, for DM is equal* to DB: * Cor. 4. 2.
therefore the gnomon CMG is equal to the rectangle AD, DB: add to each of these equals LG, which is equal to the square of CB; therefore the rectangle AD, DB, together with the square of CB, is equal to the gnomon CMG, and the figure LG: but the gnomon CMG and LG make up the whole figure CEFD, which is the square of CD; therefore the rectangle AD, DB, together with the square of CB, is equal to the square of CD. Wherefore, if a straight line, &c. Q. E. D.

PROP. VII. THEOR.

If a straight line be divided into any two parts, the squares of the whole line, and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

Let the straight line AB be divided into any two parts in the point C; the squares of AB, BC shall be equal to twice the rectangle AB, BC, together with the square of AC.

Upon AB describe* the square ADEB, and construct the figure as in the preceding propositions. * 46. 1.
Because AG is equal* to GE, add to each of them CK; * 43. 1.
therefore the whole AK is equal to the whole CE;



therefore AK, CE, are double of AK: but AK, CE are the gnomon AKF, together with the square CK; therefore the gnomon AKF, together with the square CK, is double of AK: but twice the rectangle AB, BC is double of AK, for BK is equal* to BC: therefore the gnomon AKF, together with the square CK, is equal to twice the rectangle AB, BC: to each of these equals add HF, which is equal to the square of AC; therefore the gnomon AKF, together with the squares CK, HF, is equal to twice the rectangle AB, BC, and the square of AC: but the gnomon AKF, together with the squares CK, HF, make up the whole figure ADEB and CK, which are the squares of AB and BC: therefore the squares of AB and BC are equal to twice the rectangle AB, BC, together with the square of AC. Wherefore, if a straight line, &c. Q. E. D.

* Cor. 4. 2.

PROP. VIII. THEOR.

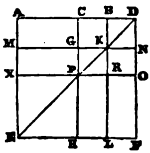
If a straight line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the square of the other part, is equal to the square of the straight line which is made up of the whole and that part.

Let the straight line AB be divided into any two parts in the point C; four times the rectangle AB, BC, together with the square of AC, shall be equal to the square of the straight line made up of AB and BC together.

In AB produced take BD equal to CB, and upon AD describe the square AEFD; and construct two figures such as in the preceding. Because CB is equal to BD, and that CB is equal* to GK, and BD to KN; therefore GK is equal to KN: for the same reason,

* 34. 1.

PR is equal to RO; and because CB is equal to BD,
and GK to KN, the rectangle CK is equal* to BN, * 36. 1.
and GR to RN: but CK is equal* to RN, because * 43. 1.
they are the complements of the parallelogram CO;
therefore, also, BN is equal to GR; therefore the four
rectangles BN, CK, GR, RN are equal to one another,
and so are quadruple of one of them, CK. Again, be-
cause CB is equal to BD, and that
BD is equal* to BK, that is, to CG; * Cor. 4. 2.
and CB equal to GK, that is,* to * Cor. 4. 2.
GP; therefore CG is equal to GP;
and because CG is equal to GP, and
PR to RO, the rectangle AG is
equal to MP, and PL to RF: but MP



is equal* to PL, because they are the complements of * 43. 1.
the parallelogram ML; wherefore AG is also equal
to RF: therefore the four rectangles AG, MP, PL, RF
are equal to one another, and so are quadruple of one
of them, AG. But it was demonstrated, that the four
CK, BN, GR, and RN are quadruple of CK: there-
fore the eight rectangles which contain the gnomon
AOH, are quadruple of AK: and because BN is a
square,* BK is equal to BD: but BD is equal to BC; * 30 Def. 1.
therefore BK is equal to BC; and because AK is the
rectangle contained by AB, BC, for BK has been
proved equal to BC; therefore four times the rect-
angle AB, BC is quadruple of AK: but the gnomon
AOH was demonstrated to be quadruple of AK;
therefore four times the rectangle AB, BC, is equal to
the gnomon AOH: to each of these equals add XH,
which is equal* to the square of AC: therefore four * Cor. 4. 2.
times the rectangle AB, BC, together with the square
of AC, is equal to the gnomon AOH and the square
XH: but the gnomon AOH and XH make up the
figure AEFD, which is the square of AD: therefore
four times the rectangle AB, BC, together with the

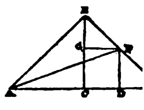
square of AC, is equal to the square of AD, that of AB and BC added together in one straight lin Wherefore, if a straight line, &c. Q. E. D.

PROP. IX. THEOR.

If a straight line be divided into two equal, and al into two unequal parts, the squares of the two unequ parts are together double of the square of half t. line, and of the square of the line between the poin of section.

Let the straight line AB be divided into two equ parts at the point C, and into two unequal parts at tl point D: the squares of AD, DB shall be togeth double of the squares of AC, CD.

* 11. 1. From the point C draw* CE at right angles to A
* 3. 1. and make* it equal to AC or CB, and join EA, El
* 31. 1. through D draw* DF parallel to CE, and through draw FG parallel to AB; and join AF. Then, becau
* 5. 1. AC is equal to CE, the angle AEC is equal* to t angle EAC; and because the angle ACE is a rig angle, the two others AEC, EAC, together make on
* 32. 1. right angle; and they have been proved equal to o another; therefore each of them is half of a right ang For the same reason each of the angles CEB, EBC half a right angle; and therefore the whole AEB is a right angle: and because the angle GEF is half a right angle, and EGF a right
* 29. 1. angle, for it is equal* to the in-



terior and opposite angle ECB; therefore the remai ing angle EFG is half a right angle; therefore t angle GEF is equal to the angle EFG, and the si
* 6. 1. EG* to the side GF. Again, because the angle at is half a right angle, and FDB a right angle, for it
* 29. 1. equal* to the interior and opposite angle ECB, thei fore the remaining angle BFD is half a right ang

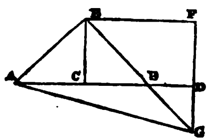
therefore the angle at B is equal to the angle BFD, and the side DF to* the side DB. And because AC * 6. 1.
is equal to CE, the square of AC is equal to the square of CE; therefore the squares of AC, CE are double of the square of AC: but the square of AE is equal* to * 47. 1.
the squares of AC, CE, because ACE is a right angle; therefore the square of EA is double of the square of AC. Again, because EG is equal to GF, the square of EG is equal to the square of GF; therefore the squares of EG, GF are double of the square of GF; but the square of EF is equal* to the squares of EG, GF; * 47. 1.
therefore the square of EF is double of the square GF; and GF is equal* to CD; therefore the square of EF * 34. 1.
is double of the square of CD: but the square of AE is likewise double of the square of AC; therefore the squares of AE, EF are double of the squares of AC, CD: but the square of AF is equal* to the squares of * 47. 1.
AE, EF, because AEF has been proved to be a right angle; therefore the square of AF is double of the squares of AC, CD: but the squares of AD, DF are equal to the square of AF, because the angle ADF is a right angle; therefore the squares of AD, DF are double of the squares of AC, CD: but DF is equal to DB; therefore the squares of AD, DB are double of the squares of AC, CD. Therefore, if a straight line, &c. Q. E. D.

PROP. X. THEOR.

If a straight line be bisected, and produced to any point, the square of the whole line thus produced, and the square of the part of it produced, are together double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to the point D; the squares of AD, DB shall be double of the squares of AC, CD.

* 11. 1. From the point C draw* CE at right angles to AB,
* 3. 1. and make* it equal to AC or CB, and join AE, EB;
* 31. 1. through E draw* EF parallel to AB, and through D
draw DF parallel to CE. And because the straight
line EF meets the parallels EC, FD, the angles CEF,
* 29. 1. EFD are equal* to two right angles; and therefore
the angles BEF, EFD are less than two right angles:
but straight lines, which with another straight line
make the interior angles upon the same side less than
* 12 Ax. two right angles, will meet* if produced far enough:
therefore EB, FD will meet, if produced towards B, D;
let them meet in G, and join AG: then, because AC
* 5. 1. is equal to CE, the angle AEC is equal* to the angle
EAC; and the angle ACE is a right angle; therefore
* 32. 1. each of the angles CEA, EAC is half a right angle;*
for the same reason, each of the angles CEB, EBC is
half a right angle; therefore AEB is a right angle.
* 15. 1. And because EBC is half a right angle, DBG is also*
half a right angle, for they are vertically opposite; but
* 29. 1. BDG is a right angle, because it is equal* to the alternate angle DCE; therefore the remaining angle DGB
is half a right angle, and is therefore equal to the
angle DBG; therefore the
* 6. 1. side BD is equal* to the side
DG. Again, because EGF is
half a right angle, and that
the angle at F is a right an-
* 34. 1. gle, because it is equal* to
the opposite angle ECD, the
remaining angle FEG is half a right angle, and equal
* 6. 1. to the angle EGF; therefore the side GF is equal* to
the side FE. And because EC is equal to CA, the
square of EC is equal to the square of CA; therefore
the squares of EC, CA are double of the square of CA:
* 47. 1. but the square of EA is equal* to the squares EC, CA;
therefore the square of EA is double of the square of
AC. Again, because GF is equal to FE, the square



of GF is equal to the square of FE; and therefore the squares of GF, FE are double of the square of EF: but the square of EG is equal* to the squares of EF, FG; therefore the square of EG is double of the square of EF: but EF is equal* to CD; therefore the square of EG is double of the square of CD: but it was demonstrated, that the square of EA is double of the square of AC; therefore the squares of AE, EG are double of the squares of AC, CD: but the square of AG is equal* to the squares of AE, EG; therefore the square of AG is double of the squares of AC, CD: but the squares of AD, DG are equal* to the square of AG; therefore the squares of AD, DG are double of the squares of AC, CD: but DG is equal to DB; therefore the squares of AD, DB are double of the squares of AC, CD. Wherefore, if a straight line, &c. Q. E. D.

* 47. 1.
* 34. 1.
* 47. 1
* 47. 1.

PROP. XI. PROB.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts, shall be equal to the square of the other part.

Let AB be the given straight line; it is required to divide it into two parts, so that the rectangle contained by the whole and one of the parts, shall be equal to the square of the other part.

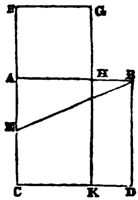
Upon AB describe* the square ABDC; bisect* AC in E, and join BE; produce CA to F, making* EF equal to EB, and upon AF describe* the square FGHA; then AB shall be divided in H, so that the rectangle AB, BH is equal to the square of AH.

* 46. 1.
* 10. 1.
* 3. 1.
* 46. 1.

Produce GH to K: and because the straight line AC is bisected in E, and produced to the point F, the rectangle CF, FA, together with the square of AE, is

* 6. 2. equal* to the square of EF: but EF is equal to EB; therefore the rectangle CF, FA, together with the square of AE, is equal to the square of EB: but the

* 47. 1. squares of BA, AE are equal* to the square of EB, because the angle EAB is a right angle; therefore the rectangle CF, FA, together with the square of AE, is equal to the squares of BA, AE: take away the square of AE, which is common to both; therefore the rectangle contained by CF, FA is equal to the square of AB: and the figure FK is the rectangle contained by CF, FA, for AF is equal to FG; and AD is the square of AB; therefore FK is equal to AD: take away the common part AK, and the remainder FH is equal to the remainder HD: but HD is the rectangle contained by AB, BH, for AB is equal to BD; and FH is the square of AH; therefore the rectangle AB, BH is equal to the square of AH. Wherefore the straight line AB is divided in H, so that the rectangle AB, BH is equal to the square of AH. Which was to be done.

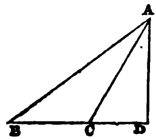


PROP. XII. THEOR.

In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle, is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.

Let ABC be an obtuse-angled triangle, having the

obtuse angle ACB, and from the point A let AD be drawn* perpendicular to BC produced: the square of AB shall be greater than the squares of AC, CB, by twice the rectangle BC, CD. * 12. 1.



Because the straight line BD is divided into two parts in the point C, the square of BD is equal* to the squares of BC, CD, and twice the rectangle BC, CD: to each of these equals add the square of DA; therefore the squares of BD, DA are equal to the squares of BC, CD, DA, and twice the rectangle BC, CD: but the square of BA is equal* to the squares of BD, DA, because the angle at D is a right angle; and the square of CA is equal* to the squares of CD, DA: therefore the square of BA is equal to the squares of BC, CA, and twice the rectangle BC, CD; that is, the square of BA is greater than the squares of BC, CA, by twice the rectangle BC, CD. Therefore, in obtuse-angled triangles, &c. Q. E. D. * 4. 2. * 47. 1. * 47. 1.

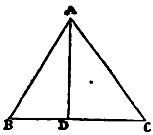
PROP. XIII. THEOR.

In every triangle, the square of the side subtending an acute angle, is less than the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the acute angle and the perpendicular let fall upon it from the opposite angle.

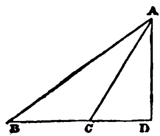
Let ABC be any triangle, and the angle at B one of its acute angles, and upon BC, one of the sides containing it, let fall the perpendicular* AD from the opposite angle: the square of the side AC, opposite * 12. 1.

to the angle B, shall be less than the squares of CB, BA, by twice the rectangle CB, BD.

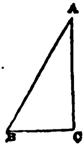
First, let AD fall within the triangle ABC; and because the straight line CB is divided into two parts in
* 7. 2. the point D, the squares of CB, BD are equal* to twice the rectangle contained by CB, BD, and the square of DC: to each of these equals add the square of AD; therefore the squares of CB, BD, DA, are equal to twice the rectangle CB, BD, and the squares of AD, DC: but the
* 47. 1. square of AB is equal* to the squares of BD, DA, because the angle BDA is a right angle; and the square of AC is equal to the squares of AD, DC: therefore the squares of CB, BA are equal to the square of AC, and twice the rectangle CB, BD; that is, the square of AC alone is less than the squares of CB, BA, by twice the rectangle CB, BD.



Secondly, let AD fall without the triangle ABC: then, because the angle at D is a right angle, the an-
* 16. 1. gle ACB is greater* than a right angle; therefore the square of AB
* 12. 2. is equal* to the squares of AC, CB, and twice the rectangle BC, CD: to each of these equals add the square of BC; therefore the squares of AB, BC are equal to the square of AC, and twice the square of BC, and twice the rectangle BC, CD: but because BD is divided into two parts in C, the rect-
* 3. 2. angle DB, BC is equal* to the rectangle BC, CD, and the square of BC; and the doubles of these are equal: therefore the squares of AB, BC are equal to the square of AC, and twice the rectangle DB, BC: therefore the square of AC alone is less than the squares of AB, BC, by twice the rectangle DB, BC.



Lastly, let the side AC be perpendicular to BC; then BC is the straight line between the perpendicular and the acute angle at B; and it is manifest that the squares of AB, BC, are equal* to the square of AC, and twice the square of BC. Therefore, in every triangle, &c. Q. E. D. * 47. 1.

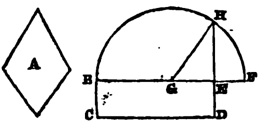


PROP. XIV. PROB.

To describe a square that shall be equal to a given rectilineal figure.

Let A be the given rectilineal figure; it is required to describe a square that shall be equal to A.

Describe* the rectangular parallelogram BCDE * 45 1.
equal to the rectilineal figure A. Then, if the sides of it, BE, ED are equal to one another, it is a square, and what was required is now done: but if they are not equal, in BE produced, take EF equal to ED, and bisect BF in G; and from the centre G, at the distance GB, or GF, describe the semicircle BHF, and produce DE to meet the circumference in H; then the square described on EH shall be equal to the given rectilineal figure A. Join GH; and because the straight line BF is divided into two equal parts in the point G, and into two unequal at E, the rectangle BE, EF, together with the square of EG, is equal* to the square of GF: but GF is equal to * 5. 2.
GH; therefore the rectangle BE, EF, together with the square of EG, is equal to the square of GH; but the squares of HE, EG are equal* to the square of * 47. 1.
GH: therefore the rectangle BE, EF, together with



the square of EG, is equal to the squares of HE, EG: take away the square of EG, which is common to both; therefore the rectangle contained by BE, EF, is equal to the square of EH: but the rectangle contained by BE, EF is the parallelogram BD, because EF is equal to ED; therefore BD is equal to the square of EH;
* Const. but BD is equal* to the rectilineal figure A; therefore the square of EH is equal to the rectilineal figure A; wherefore a square has been made equal to the given rectilineal figure A, viz. the square described upon EH. Which was to be done.

THE

ELEMENTS OF EUCLID.

BOOK III.

DEFINITIONS.

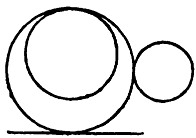
I.

Equal circles are those of which the diameters are equal, or from the centres of which the straight lines to the circumferences are equal.

'This is not a definition, but a theorem, the truth of 'which is evident; for, if the circles be applied to one 'another, so that their centres coincide, the circles 'must likewise coincide, since the straight lines from 'the centres are equal.'

II.

A straight line is said to *touch* a circle, when it meets the circle, and being produced does not cut it.

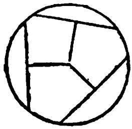


III.

Circles are said to *touch* one another, which meet, but do not cut one another.

IV.

Straight lines are said to be ***equally distant from the centre*** of a circle, when the perpendiculars drawn to them from the centre are equal.



V.

And the straight line on which the greater perpendicular falls, is said to be ***farther from the centre.***

VI.

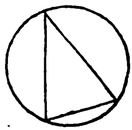
A ***segment of a circle*** is the figure contained by a straight line and the circumference it cuts off.

VII.

" The ***angle of a segment*** is that which is contained by " the straight line and the circumference."

VIII.

An ***angle in a segment*** is the angle contained by two straight lines drawn from any point in the circumference of the segment, to the extremities of the straight line which is the base of the segment.

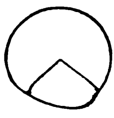


IX.

And an angle is said to insist or ***stand upon*** the circumference intercepted between the straight lines that contain the angle.

X.

A ***sector*** of a circle is the figure contained by two straight lines drawn from the centre, and the circumference between them.



XI.

Similar segments of circles are those in which the angles are equal, or which contain equal angles.

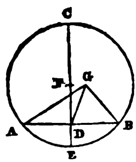
PROP. I. PROB.

To find the centre of a given circle.

Let ABC be the given circle; it is required to find its centre.

Draw any chord AB, and bisect* it in D; from the point D draw* DC at right angles to AB, and produce CD to E, and bisect CE in F: the point F shall be the centre of the circle ABC. * 10. 1. * 11. 1.

For, if it be not, let, if possible, G be the centre, and join GA, GD, GB: then, because DA is equal to DB, and DG common to the two triangles ADG, BDG, the two sides AD, DG are equal to the two BD, DG, each to each; and the base GA is equal to the base GB, because they are drawn from the centre G:[a] therefore the angle ADG is equal* to the angle GDB: but when a straight line, standing upon another straight line, makes the adjacent angles equal to one another, each of the angles is a right* angle: therefore the angle GDB is a right angle: but FDB is likewise a right angle; wherefore the angle FDB is equal to the angle GDB, the greater to the less, which is impossible: therefore G is not the centre of the circle ABC. In the same manner it can be shown, * 8. 1. * 10 Def. 1.



[a] Whenever the expression "straight lines from the centre," or "drawn from the centre," occurs, it is to be understood that they are drawn to the circumference.

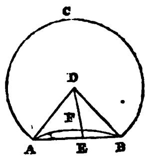
that no other point, which is not in the line CE, is the centre; the centre is therefore in CE, and any other point than F divides CE into unequal parts, and cannot be the centre; therefore F is the centre of the circle ABC. Which was to be found.

COR. From this it is manifest, that if in a circle a straight line bisect another at right angles, the centre of the circle is in the line which bisects the other.

PROP. II. THEOR.

If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

Let ABC be a circle, and A, B any two points in the circumference; the straight line drawn from A to B shall fall within the circle.



For, if it do not, let it fall, if possible, without, as AEB; find* D the centre of the circle ABC, and join DA, DB. In the circumference AB take any point F; join DF, and produce it to meet AB in E: then, because DA is equal to DB, the angle DAB is equal* to the angle DBA; and because AE, a side of the triangle DAE, is produced to B, the angle DEB is greater* than the angle DAE; but DAE was proved equal to the angle DBE: therefore the angle DEB is greater than the angle DBE; but the greater angle is subtended by the greater side;* therefore DB is greater than DE: but DB is equal to DF; wherefore DF is greater than DE, the less than the greater, which is impossible: therefore the straight line drawn from A to B does not fall without the circle. In the same manner, it may be demonstrated that it does not

* 1. 3.
* 5. 1.
* 16. 1.
* 19. 1.

fall upon the circumference; therefore it falls within it. Wherefore, if any two points, &c. Q. E. D.

PROP. III. THEOR.

If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles; and, if it cut it at right angles, it shall bisect it.

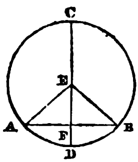
Let ABC be a circle; and let CD, a straight line drawn through the centre, bisect any straight line AB, which does not pass through the centre, in the point F: it shall cut AB at right angles.

Take* E the centre of the circle, and join EA, EB; then, because AF is equal to FB, and FE common to the two triangles AFE, BFE, the two sides AF, FE are equal to the two BF, FE, and the base EA is equal to the base EB; therefore the angle AFE is equal* to the angle BFE: but when a straight line, standing upon another straight line, makes the adjacent angles equal to one another, each of them is a right* angle: therefore each of the angles AFE, BFE is a right angle; wherefore the straight line CD, drawn through the centre bisecting another, AB, that does not pass through the centre, cuts the same at right angles.

* 1. 3.

* 8. 1.

* 10 Def. 1.



But let CD cut AB at right angles; CD shall also bisect it, that is, AF shall be equal to FB.

The same construction being made, because EA, EB from the centre are equal to one another, the angle EAF is equal* to the angle EBF; and the right angle AFE is equal to the right angle BFE: therefore, the two triangles EAF, EBF, have two angles in the one

* 5. 1.

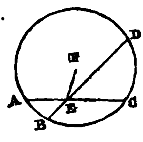
equal to two angles in the other, and the side EF, which is opposite to one of the equal angles in each, is common to both; therefore the other sides are
* 26. 1. *equal; and therefore AF is equal to FB. Wherefore, if a straight line, &c. Q. E. D.

PROP. IV. THEOR.

If in a circle two straight lines cut one another, which do not both pass through the centre, they do not bisect each other.

Let ABCD be a circle, and AC, BD two straight lines in it which cut one another in the point E, and do not both pass through the centre: AC, BD shall not bisect one another.

For, if it is possible, let AE be equal to EC, and BE to ED; then, if one of the lines pass through the centre, it is plain that it cannot be bisected by the other which does not pass through the centre: but if neither of them
* 1. 3. pass through the centre, take* F the centre of the circle, and join EF: and because FE, a straight line through the centre, bisects another



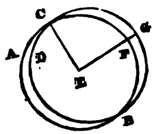
AC which does not pass through the centre, it cuts it
* 3. 3. at right* angles; wherefore FEA is a right angle. Again, because the straight line FE bisects the straight line BD, which does not pass through the centre, it
* 3. 3. cuts it at right* angles; wherefore FEB is a right angle: but FEA was proved to be a right angle; therefore FEA is equal to the angle FEB, the less to the greater, which is impossible: therefore AC, BD do not bisect one another. Wherefore, if in a circle, &c. *Q. E. D.*

PROP. V. THEOR.

If two circles cut one another, they shall not have the same centre.

Let the two circles ABC, CDG cut one another in the points B, C; they shall not have the same centre.

For, if it be possible, let E be their centre: join EC, and draw any straight line EFG, meeting them in F and G: and because E is the centre of the circle ABC, CE is equal to EF. Again, because E is the centre of the circle CDG, CE is equal to EG: but CE was proved to be equal to EF; therefore EF is equal to EG, the less to the greater, which is impossible: therefore E is not the centre of the circles ABC, CDG. Wherefore, if two circles, &c. Q. E. D.

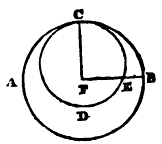


PROP. VI. THEOR.

If one circle touch another internally, they shall not have the same centre.

Let the circle CDE touch the circle ABC internally in the point C: they shall not have the same centre.

For, if they have, let it be F; join FC, and draw any straight line FEB, meeting them in E and B; and because F is the centre of the circle ABC, FC is equal to FB: also, because F is the centre of the circle CDE, FC is equal to FE: but FC was proved equal to FB; therefore FE is equal to FB, the less to the greater, which is impossible:



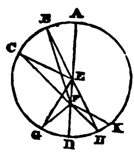
therefore F is not the centre of the circles ABC, CDE. Wherefore, if one circle, &c. Q. E. D.

PROP. VII. THEOR.

If any point be taken in the diameter of a circle which is not the centre, of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is, and the other part of that diameter is the least; and, of any others, that which is nearer to the line which passes through the centre is always greater than one more remote: and from the same point there can be drawn two, and only two, straight lines to the circumference, that are equal to one another, one upon each side of the diameter.

Let ABCD be a circle, and AD its diameter, in which let any point F be taken which is not the centre: let E be the centre; of all the straight lines FB, FC, FG, &c. that can be drawn from F to the circumference, FA, that in which the centre is, shall be the greatest, and FD, the other part of the diameter AD, shall be the least: and of the others, FB, the nearer to FA, shall be greater than FC the more remote, and FC than FG.

Join BE, CE, GE; and because two sides of a tri-
* 20. 1. angle are greater* than the third, therefore FE, EB are greater than FB; but EA is equal to EB; therefore FE, EA, that is, FA, is greater than FB. Again, because EB is equal to EC, and EF common to the triangles FBE, FCE, the two sides FE, EB are equal to the two FE, EC, each to each; but the angle FEB is greater than the angle FEC; therefore the base FB is
* 24. 1. greater* than the base FC. In like manner it may be proved that FC is greater than FG. Again, be-



cause EF, FG are greater* than EG, and EG is equal to ED; therefore EF, FG are greater than ED: take away the common part EF, and the remainder FG is greater than the remainder FD. Therefore, FA is the greatest, and FD the least of all the straight lines from F to the circumference; and FB is greater than FC, and FC than FG. * 20. 1.

Also, there can be drawn only two equal straight lines from the point F to the circumference, one upon each side of the diameter. At the point E, in the straight line EF, make* the angle FEH equal to the angle FEG, and join FH: then, because EG is equal to EH, and EF common to the two triangles FEG, FEH; the two sides FE, EG are equal to the two FE, EH; and the angle FEG is equal to the angle FEH; therefore the base FG is equal* to the base FH: but, besides FH, no other straight line can be drawn from F to the circumference equal to FG; for, if there can, let it be FK; and because FK is equal to FG, and FG to FH, FK is equal to FH; that is, a line nearer to that which passes through the centre, is equal to one which is more remote; which has been proved to be impossible. Therefore, if any point be taken, &c. Q. E. D. * 23. 1. * 4. 1.

PROP. VIII. THEOR.

If any point be taken without a circle, and straight lines be drawn from it to the circumference, whereof one passes through the centre; of those which fall upon the concave circumference, the greatest is that which passes through the centre; and of the rest, that which is nearer to the one through the centre is always greater than one more remote: but of those which fall upon the convex circumference, the least is that between the point without the circle, and the diameter; and of the rest, that which is nearer to the least is always less than one more remote: and only two equal

straight lines can be drawn from the point to the circumference, one upon each side of the line through the centre.

Let ABC be a circle, and D any point without it, from which the straight lines DA, DE, DF, DC, are drawn to the circumference, whereof DA passes through the centre. Of those which fall upon the concave part of the circumference AEFC, the greatest shall be DA, which passes through the centre; and any line nearer to it shall be greater than one more remote, viz. DE shall be greater than DF, and DF than DC: but of those which fall upon the convex circumference HLKG, the least shall be DG between the point D and the diameter GA; and any line nearer to it shall be less than one more remote, viz. DK less than DL, and DL than DH.

* 1. 3. Take* M the centre of the circle ABC, and join ME, MF, MC, MK, ML, MH. Because AM is equal to EM, add MD to each, therefore AD is equal to EM, MD; but EM, MD are greater* than ED; therefore also AD is greater than ED. Again, because EM is equal to FM, and MD common to the triangles EMD, FMD; the two sides EM, MD are equal to FM, MD, each to each; but the angle EMD is greater than the angle FMD; therefore the base ED is greater* than the base FD. In like manner it may be shown that FD is greater than CD: therefore DA is the greatest; and DE greater than DF, and DF than DC. And because MK, KD are greater* than MD, and MK is equal to MG, the remainder KD is greater* than the remainder GD, that is, GD is less than KD: and because MK, DK are drawn to the point K within the

* 20. 1.
* 24. 1.
* 20. 1.
* 5 Ax.

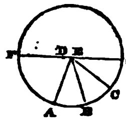
triangle MLD from M, D, the extremities of its side MD, therefore MK, KD are less* than ML, LD; * 21. 1.
but MK is equal to ML; therefore the remainder DK is less* than the remainder DL. In like manner it * 5 Ax.
may be proved, that DL is less than DH: therefore DG is the least, and DK less than DL, and DL than DH. Also, there can be drawn only two equal straight lines from the point D to the circumference, one upon each side of the line through the centre. At the point M, in the straight line MD, make* the angle DMB * 23. 1.
equal to the angle DMK, and join DB: and because MK is equal to MB, and MD common to the triangles KMD, BMD, the two sides KM, MD are equal to the two BM, MD, each to each; and the angle KMD is equal to the angle BMD; therefore the base DK is equal* to the base DB: but, besides DB, there can be * 4. 1.
no straight line drawn from D to the circumference equal to DK: for, if there can, let it be DN; and because DK is equal to DN, and also to DB; therefore DB is equal to DN, that is, a line nearer to the least equal to one more remote, which has been proved to be impossible. If, therefore, any point, &c. Q. E. D.

PROP. IX. THEOR.

If a point be taken within a circle, from which there fall more than two equal straight lines upon the circumference, that point is the centre of the circle.

Let the point D be taken within the circle ABC, from which to the circumference there fall more than two equal straight lines, viz. DA, DB, DC: the point D shall be the centre of the circle.

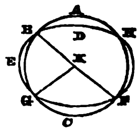
For, if not, let E be the centre; join DE, and produce it to the circumference in F, G; then FG is a diameter of the circle ABC: and because in FG, the diameter of the circle ABC, the point D is taken, which is not the centre, DG is the greatest line from



it to the circumference, and DC
* 7. 3. greater* than DB, and DB than DA: but this is impossible, because DA, DB, DC are equal by hypothesis; therefore E is not the centre of the circle ABC. In like manner, it may be demonstrated, that no other point but D is the centre; therefore D is the centre. Wherefore, if a point be taken, &c. Q. E. D.

PROP. X. THEOR.

One circumference of a circle cannot cut another in more than two points.



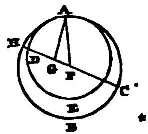
If it be possible, let the circumference FAB cut the circumference DEF in more than two points, viz. in
* 1. 3. B, G, F; take* K, the centre of the circle ABC, and join KB, KG, KF: therefore KB, KG, KF are all equal to each other: and because within the circle DEF the point K is taken, from which more than two equal straight lines KB, KG, KF fall on the circumference
* 9. 3. DEF, the point K is* the centre of the circle DEF: but K is also the centre of the circle ABC; therefore the same point is the centre of two circles that cut
* 5. 3. one another, which is impossible.* Therefore the circumference of a circle cannot cut that of another in more than two points. Q. E. D.

PROP. XI. THEOR.

If one circle touch another internally in any point, the straight line which joins their centres being produced, shall pass through that point.

Let the circle ADE touch the circle ABC inter-

nally in the point A, and let F be the centre of the circle ABC, and G the centre of the circle ADE: the straight line which joins the centres F, G, being produced, shall pass through the point A.



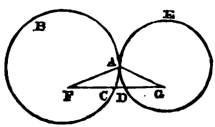
For, if not, let it fall otherwise, if possible, as FGDH, and join AF, AG: and because FG, GA are greater* than FA, and FA is equal to FH; therefore FG, GA are greater than FH: take away the common part FG; therefore the remainder GA is greater* than the remainder GH: but GA is equal to GD; therefore GD is greater than GH, the less than the greater, which is impossible. Therefore the straight line which joins the points F, G being produced, cannot fall otherwise than upon the point A, that is, it must pass through it. Therefore, if one circle, &c. Q. E. D. * 20. 1 * 5 Ax.

PROP. XII. THEOR.

If two circles touch each other externally in any point, the straight line which joins their centres shall pass through that point.

Let the two circles ABC, ADE touch each other externally in the point A; and let F be the centre of the circle ABC, and G the centre of ADE: the straight line which joins the points F, G, shall pass through A, the point of contact.

For, if not, let it pass otherwise, if possible, as FCDG, and join FA, AG: and because F is the centre of the circle ABC, FA is equal to FC: also, because G is the centre of the circle ADE, GA is equal to GD: therefore FA, AG are equal* to FC, DG: wherefore the * 2 Ax.

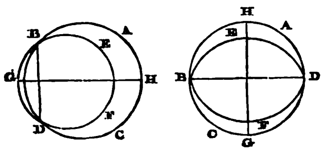


whole FG is greater than FA, AG; but this is imp
* 20. 1. sible, for FG is less* than FA, AG. Therefore
straight line which joins the points F, G, cannot p
otherwise than through A, the point of contact, that
it must pass through it. Therefore, if two circles,
Q. E. D.

PROP. XIII. THEOR.

One circle cannot touch another in more points than o whether it touch it on the inside or on the outside

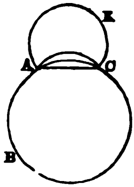
For, if it be possible, let the circle EBF touch circle ABC in more points than one, and first on
* 10. 11. inside, in the points B, D: join BD, and draw* (
1. bisecting BD at right angles: then, because the po B, D are in the circumference of each of the circ



* 2. 3. the straight line BD falls within each* of them:
* Cor. 1. 3. their centres are* in the straight line GH, which sects BD at right angles: therefore GH passes thro
* 11. 3. the point of contact;* but it does not pass through because the points B, D are without the straight l GH, which is absurd: therefore one circle can touch another on the inside in more points than one

Nor can one circle touch another on the outside more than one point: for, if it be possible, let the ci ACK touch the circle ABC in the points A, C, join AC: then, because the two points A, C are in circumference of the circle ACK, the straight line
* 2. 3. which joins them falls within* the circle ACK:

the circle ACK is without the circle ABC; and therefore the straight line AC is without this last circle; but, because the points A, C are in the circumference of the circle ABC, the straight line AC must be within* the same circle, which is absurd: therefore one circle cannot touch another on the outside in more than one point: and it has been proved, that one circle cannot touch another on the inside in more points than one. Therefore, one circle, &c. Q. E. D.



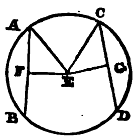
* 2. 3.

PROP. XIV. THEOR.

Equal straight lines in a circle are equally distant from the centre; and those which are equally distant from the centre, are equal to one another.

Let the straight lines AB, CD, in the circle ABDC, be equal to one another; they shall be equally distant from the centre.

Take* E the centre of the circle ABDC, and from it draw* EF, EG perpendiculars to AB, CD, and join AE, EC: then, because the straight line EF, passing through the centre, cuts the straight line AB, which does not pass through the centre, at right angles, it also bisects* it: therefore AF is equal to FB, and AB double of AF. For the same reason CD is double of CG; but AB is equal to CD; therefore AF is equal to CG: and because AE is equal to CE, the square of AE is equal to the square of CE: but the squares of AF, FE are equal* to the square of AE, because the angle AFE is a right angle; and for a like reason, the squares of CG, GE are equal to the square of CE: therefore the squares



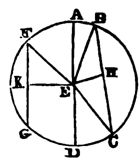
* 1. 3.
* 12. 1.
* 3. 3.
* 47. 1.

of AF, FE are equal to the squares of CG, GE; but the square of AF is equal to the square of CG, because AF is equal to CG; therefore the remaining square of EF is equal to the remaining square of GE, and the straight line EF is therefore equal to EG: but straight lines in a circle are said to be equally distant from the centre, when the perpendiculars drawn to them from
* 4 Def. 3. the centre are equal:* therefore AB, CD are equally distant from the centre.

Next, if the straight lines AB, CD be equally distant from the centre, that is, if FE be equal to EG; AB shall be equal to CD. For, the same construction being made, it may, as before, be demonstrated, that AB is double of AF, and CD double of CG, and that the squares of EF, FA are equal to the squares of EG, GC; but the square of EF is equal to the square of EG, because EF is equal to EG; therefore the remaining square of AF is equal to the remaining square of CG; and the straight line AF is therefore equal to CG; but AB was proved to be double of AF, and CD double
* 6 Ax. of CG; wherefore AB is equal* to CD. Therefore, equal straight lines, &c. Q. E. D.

PROP. XV. THEOR.

The diameter is the greatest straight line in a circle: and, of all others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.



Let ABCD be a circle, of which the diameter is AD, and the centre E; and let BC be nearer to the centre than FG; AD shall be greater than any straight line BC which is not a diameter, and BC shall be greater than FG.

From E the centre draw EH, EK perpendiculars to

BC, FG, and join EB, EC, EF; and because AE is equal to EB, and ED to EC, therefore AD is equal to EB, EC: but EB, EC are greater* than BC; wherefore AD is also greater than BC. * 20. 1.

And, because BC is nearer to the centre than FG, EH is less* than EK: but, as was demonstrated in the preceding, BC is double of BH, and FG double of FK, and the squares of EH, HB are equal to the squares of EK, KF; but the square of EH is less than the square of EK, because EH is less than EK; therefore the square of BH is greater than the square of FK, and the straight line BH greater than FK; and therefore BC is greater than FG. * 5 Def. 3.

Next, let BC be greater than FG; BC shall be nearer to the centre than FG, that is,* the same construction being made, EH shall be less than EK. Because BC is greater than FG, BH is greater than FK: and the squares of BH, HE are equal to the squares of FK, KE; but the square of BH is greater than the square of FK, because BH is greater than FK; therefore the square of EH is less than the square of EK, and the straight line EH less than EK; therefore BC is nearer* to the centre than FG. Wherefore, the diameter, &c. Q. E. D. * 5 Def. 3. * 5 Def 3.

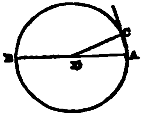
PROP. XVI. THEOR.

The straight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and no straight line can be drawn from the extremity between that straight line and the circumference, so as not to cut the circle; or, which is the same thing, no straight line can make so great an acute angle with the diameter at its extremity, or so small an angle with the straight line which is at right angles to it, as not to cut the circle.

Let ABC be a circle, the centre of which is D, and

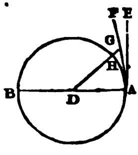
the diameter AB: the straight line drawn at right angles to AB from its extremity A, shall fall without the circle.

For, if it does not, let it fall, if possible, within the circle, as AC, and draw DC to the point C, where it meets the circumference: and because DA is equal to DC, the angle DCA is equal * to the angle DAC; but DAC is a right angle, therefore DCA is a right angle, and the angles DAC, DCA are therefore equal to two right angles; which is impossible:* therefore the straight line drawn from A at right angles to BA, does not fall within the circle. In the same manner, it may be demonstrated, that it does not fall upon the circumference; therefore it must fall without the circle, as AE.*



* 5. 1.
* 17. 1.
* See fig. 2.

Also, between the straight line AE and the circumference no straight line can be drawn from the point A which does not cut the circle. For, if possible, let AF be between them, and from the point D draw* DG perpendicular to AF, and let it meet the circumference in H: and because DGA is a right angle, and DAG less* than a right angle; therefore DA is greater* than DG: but DA is equal to DH; therefore DH is greater than DG, the less than the greater, which is impossible: therefore no straight line can be drawn from the point A, between AE and the circumference, which does not cut the circle; or, which amounts to the same thing, however great an acute angle a straight line makes with the diameter at the point A, or however small an angle it makes with AE, the circumference must pass between that straight line and the perpendicular AE. "And this is all that



* 12. 1.
* 17. 1.
* 19. 1.

is to be understood, when, in the Greek text and translations from it, the angle of the semicircle is said to be greater than any acute rectilineal angle, and the remaining angle less than any rectilineal angle."

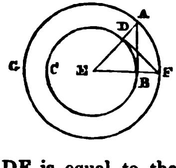
COR. From this it is manifest, that the straight line which is drawn at right angles to the diameter of a circle from the extremity of it, touches the circle; and that it touches it only in one point, because, if it did meet the circle in two, it would fall within it.* "Also, it is evident, that there can be but one straight line which touches the circle in the same point." * 2. 3.

PROP. XVII. PROB.

To draw a straight line from a given point, either without or in the circumference, which shall touch a given circle.

First, let A be a given point *without* the given circle BCD, it is required to draw a straight line from A, which shall touch the circle.

Find* E the centre of the circle, and draw AE, cutting the circle in D; and from the centre E, at the distance EA, describe the circle AFG; from the point D draw* DF at right angles to EA, and draw EBF, AB: then AB shall touch the circle BCD. * 1. 3. * 11. 1.



Because E is the centre of the circles BCD, AFG, EA is equal to EF, and ED to EB; therefore the two sides AE, EB are equal to the two FE, ED, each to each, and they contain the angle at E common to the two triangles AEB, FED; therefore the base DF is equal to the base AB, and the triangle FED to the triangle AEB, and the other angles to the other angles:* therefore the angle EBA is equal to the angle EDF: but EDF is a right angle, wherefore EBA is a right angle: and * 4. 1.

EB is drawn from the centre: but a straight line drawn from the extremity of a diameter, at right angles
* Cor. 16. 3. to it, touches* the circle; therefore AB touches the circle; and it is drawn from the given point A. Which was to be done.

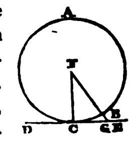
But if the given point be *in* the circumference of the circle, as the point D, draw DE to the centre E, and
* Cor. 16. 3. DF at right angles to DE; DF touches* the circle.

PROP. XVIII. THEOR.

If a straight line touch a circle, the straight line drawn from the centre to the point of contact, shall be perpendicular to the line touching the circle

Let the straight line DE touch the circle ABC in the point C; take the centre F, and draw the straight line FC: then FC shall be perpendicular to DE.

For, if it be not, from the point F draw FBG perpendicular to DE; and because FGC is a right angle,
* 17. 1. FCG is* an acute angle; and to the greater angle the
* 19. 1. greater* side is opposite: therefore FC is greater than FG; but FC is equal to FB; therefore FB is greater than FG, the less than the greater, which is impossible: therefore FG is not perpendicular to DE. In the same manner it may be proved, that no other is perpendicular to it besides FC, that is, FC is perpendicular to DE. Therefore, if a straight line, &c. Q. E. D.



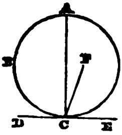
PROP. XIX. THEOR.

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

Let the straight line DE touch the circle ABC in C,

and from C let CA be drawn at right angles to DE; the centre of the circle shall be in CA.

For if not, let F be the centre, if possible, and join FC. Because DE touches the circle ABC, and FC is drawn from the centre to the point of contact, FC is perpendicular* to DE: * 18. 3.



therefore FCE is a right angle: but ACE is also a right angle; therefore the angle FCE is equal to the angle ACE, the less to the greater, which is impossible: therefore F is not the centre of the circle ABC. In the same manner, it may be proved, that no other point which is not in CA, is the centre; that is, the centre is in CA. Therefore, if a straight line, &c. Q. E. D.

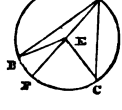
PROP. XX. THEOR.

The angle at the centre of a circle is double of the angle at the circumference, upon the same base, that is, upon the same part of the circumference.

Let ABC be a circle, and BEC an angle at the centre, and BAC an angle at the circumference, which have the same circumference BC for their base; the angle BEC shall be double of the angle BAC.

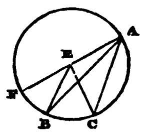
First, let E the centre of the circle be within the angle BAC, and join AE, and produce it to F: then, because EA is equal to EB, the angle EAB is equal* to the angle EBA; therefore the angles EAB, EBA are double of the angle EAB; but the angle BEF is equal* to the angles EAB, EBA; therefore also the angle BEF is double of the angle EAB: for the same reason, the angle FEC is double of the angle EAC: therefore

* 5. 1.
* 32. 1.



the whole angle BEC is double of the whole angle BAC.

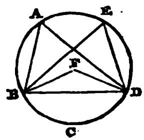
Again, let E the centre of the circle be without the angle BAC. It may be demonstrated, as in the first case, that the angle FEC is double of the angle FAC, and that FEB, a part of the first, is double of FAB, a part of the other; therefore the remaining angle BEC is double of the remaining angle BAC. Therefore the angle at the centre, &c. Q. E. D.



PROP. XXI. THEOR.

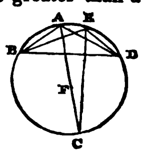
The angles in the same segment of a circle are equal to one another.

Let ABCD be a circle, and BAD, BED angles in the same segment BAED: the angles BAD, BED shall be equal to one another.



First, let the segment BAED be greater than a semicircle; find F the centre of the circle, and join BF, FD. Then, because the angle BFD is at the centre, and the angle BAD at the circumference, and that they have the same part of the circumference, viz. BCD for
* 20. 3. their base; therefore the angle BFD is double* of the angle BAD: for the same reason, the angle BFD is double of the angle BED: therefore the angle BAD
* 7 Ax. is equal* to the angle BED.

But, let the segment BAED be not greater than a semicircle, then, also, the angles BAD, BED shall be equal to one another. Draw AF to the centre, and produce it to C, and join CE: therefore the segment BADC is greater than a semicircle; and the angles in it, BAC,



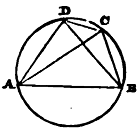
BEC are equal, by the first case. For the same reason, because CBED is greater than a semicircle, the angles CAD, CED are equal: therefore the whole angle BAD is equal* to the whole angle BED. Wherefore, the angles in the same segment, &c. Q. E. D. * 2 Ax.

PROP. XXII. THEOR.

The opposite angles of any quadrilateral figure inscribed in a circle, are together equal to two right angles.

Let ABCD be a quadrilateral figure in the circle ABCD; any two of its opposite angles shall together be equal to two right angles.

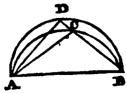
Join AC, BD; and because the three angles of every triangle are equal* to two right angles, the three angles of the triangle CAB, viz. the angles CAB, ABC, BCA are equal to two right angles: but the angle CAB is equal* to the angle CDB, because they are in the same segment BADC, and the angle ACB is equal to the angle ADB, because they are in the same segment ADCB: therefore the two angles CAB, ACB are together equal* to the whole angle ADC; to each of these equals add the angle ABC; therefore the angles ABC, CAB, BCA are equal* to the angles ABC, ADC: but ABC, CAB, BCA are equal to two right angles; therefore the angles ABC, ADC are also equal* to two right angles. In like manner, the angles BAD DCB may be proved to be equal to two right angles. Therefore, the opposite angles, &c. Q. E. D. * 32. 1. * 21. 3. * 2 Ax. * 2 Ax. * 1 Ax.



PROP. XXIII. THEOR.

Upon the same straight line, and upon the same side of it, there cannot be two similar segments of circles, not coinciding with one another.

If it be possible, upon the same straight line AB, and upon the same side of it, let ACB, ADB be two similar segments of circles not coinciding with one another; then, because the circle ACB cuts the circle ADB in the two points A, B, they cannot cut one another in any other
* 10. 3. point;* one of the segments must therefore fall within the other; let ACB fall within ADB, and draw the straight line BCD, and join CA, DA: and because the
* Hyp. segment ACB is similar* to the segment ADB, and
* 11 Def. 3. that similar segments of circles contain equal* angles; the angle ACB is equal to the angle ADB; that is, the exterior angle ACB of the triangle ACD is equal to the interior and opposite angle ADC, which is im-
* 16. 1. possible.* Therefore, there cannot be two similar segments of circles upon the same side of the same line, which do not coincide. Q. E. D.

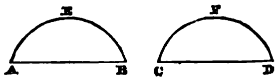


PROP. XXIV. THEOR.

Similar segments of circles upon equal straight lines, are equal to one another.

Let AEB, CFD be similar segments of circles upon the equal straight lines AB, CD; the segment AEB shall be equal to the segment CFD.

For, if the segment AEB be applied to the segment CFD, so that the point A may be on C, and the straight line AB upon CD, the point B shall coincide with the point D, because AB is equal to CD: therefore the straight line
* 23. 3. AB coinciding with CD, the segment AEB must*



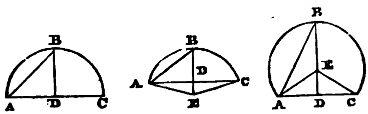
coincide with the segment CFD, and therefore is equal to it.* Wherefore, similar segments, &c. Q. E. D. * 8 Ax.

PROP. XXV. PROB.

A segment of a circle being given, to describe the circle of which it is the segment.

Let ABC be the given segment of a circle; it is required to describe the circle of which it is the segment.

Bisect* AC in D, and from the point D draw* DB at right angles to AC, and join AB. First, let the angles ABD, BAD be equal to one another; then the straight line DB is equal* to DA, and also to DC, since AC is bisected in D; and because the three straight lines DA, DB, DC, are all equal, therefore D is the centre of the circle.* From the centre D, at the distance of any of the three DA, DB, DC, describe a circle; this shall pass through the other points: and the circle of which ABC is a segment is described: and because the centre D is in AC, the * 10. 1. * 11. 1. * 6. 1. * 9. 3.



segment ABC is a semicircle. But if the angles ABD, BAD are not equal to one another, at the point A, in the straight line AB make* the angle BAE equal to the angle ABD, and produce BD, if necessary, to meet AE in E, and join EC: then, because the angle ABE is equal to the angle BAE, the straight line BE is equal* to EA: and because AD is equal to DC, and DE common to the triangles ADE, CDE, the two * 23. 1. * 6. 1.

sides AD, DE are equal to the two CD, DE, each to each; and the angle ADE is equal to the angle CDE,
* Const. for each of them is a right* angle; therefore the base
* 4. 1. AE is equal* to the base EC: but EA was proved to
* 1 Ax. be equal to EB, wherefore also EB is equal* to EC: and the three straight lines EA, EB, EC are therefore
* 9. 3. equal to one another; therefore* E is the centre of the circle. From the centre E, at the distance of any of the three EA, EB, EC, describe a circle; this shall pass through the other points; and the circle, of which ABC is a segment, is described: and it is evident, that if the angle ABD be greater than the angle BAD, the centre E falls without the segment ABC, which therefore is less than a semicircle: but if the angle ABD be less than BAD, the centre E falls within the segment ABC, which is therefore greater than a semicircle: therefore, a segment of a circle being given, the circle is described of which it is a segment. Which was to be done.

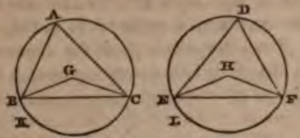
PROP. XXVI. THEOR.

In equal circles, equal angles stand upon equal circumferences, whether they be at the centres or circumferences.

Let ABC, DEF be equal circles, and let the angles BGC, EHF at their centres be equal, or BAC, EDF at their circumferences, be equal to each other: the circumference BKC shall be equal to the circumference ELF.

Join BC, EF; and because the circles ABC, DEF are equal, the straight lines drawn from their centres
* 1 Def. 3. are equal:* therefore the two sides BG, GC are equal
* Hyp. to the two EH, HF; and the angle at G is equal* to
* 4. 1. the angle at H; therefore the base BC is equal* to
* Hyp. the base EF: and because the angle at A is equal* to

the angle at D, the segment BAC is similar* to the *11 Def. 3.
segment EDF; and they are upon equal straight lines BC, EF; but similar segments of circles upon equal straight lines, are equal* to one another; therefore the *24. 3.
segment BAC is equal to the segment EDF: but the



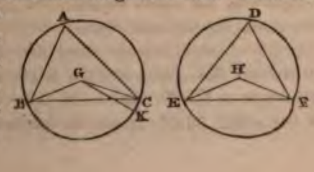
whole circle ABC is equal* to the whole DEF; there- *Hyp.
fore the remaining segment BKC is equal* to the re- *3 Ax.
maining segment ELF, and the circumference BKC to the circumference ELF. Wherefore, in equal circles, &c. Q. E. D.

PROP. XXVII. THEOR.

In equal circles, the angles which stand upon equal circumferences are equal to one another, whether they be at the centres or circumferences.

Let ABC, DEF be equal circles, and let the angles BGC, EHF at their centres, or BAC, EDF at their circumferences, stand upon the equal parts BC, EF of the circumferences: the angle BGC shall be equal to the angle EHF, and the angle BAC to the angle EDF.

If the angle BGC be equal to the angle EHF, it is manifest* that the angle BAC is also equal to EDF: *20. 3.
but, if not, one of them must be greater than the other. Let BGC be the greater, and at the point G, in the straight line BG, make* the angle BGK *equal to the* *23. 1.



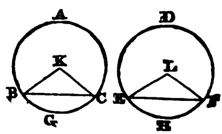
angle EHF; but equal angles stand upon equal cir-
* 26. 3. cumferences,* when they are at the centre; therefore the circumference BK is equal to the circumference
* Hyp. EF: but EF is equal* to BC; therefore also BK is
* 1 Ax. equal* to BC, the less to the greater, which is impossible: therefore the angle BGC is not unequal to the angle EHF; that is, it is equal to it: but the angle at
* 20. 3. A is half* of the angle BGC, and the angle at D half
* 7 Ax. of the angle EHF: therefore the angle at A is equal* to the angle at D. Wherefore, in equal circles, &c. Q. E. D.

PROP. XXVIII. THEOR.

In equal circles, equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less.

Let ABC, DEF be equal circles, and BC, EF equal straight lines in them, which cut off the two greater circumferences BAC, EDF, and the two less BGC, EHF: the greater BAC shall be equal to the greater EDF, and the less BGC to the less EHF.

* 1. 3. Take* K, L, the centres of the circles, and join BK, KC, EL, LF: and because the circles are equal, the
* 1 Def. 3. straight lines from their centres are* equal; therefore BK, KC are equal to EL, LF; and the base BC is
* Hyp. equal* to the base EF; therefore the angle BKC is
* 8. 1. equal* to the angle ELF: but equal angles stand upon



* 26. 3. equal* circumferences, when they are at the centres; therefore the circumference BGC is equal to the cir-
* Hyp. cumference EHF: but the whole circle ABC is equal* to the whole EDF; therefore the remaining part of the

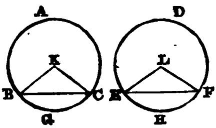
circumference, viz. BAC, is equal* to the remaining * 3 Ax.
part EDF. Therefore, in equal circles, &c. Q. E. D.

PROP. XXIX. THEOR.

In equal circles equal circumferences are subtended by equal straight lines.

Let ABC, DEF be equal circles, and let the circumferences BGC, EHF also be equal; and join BC, EF: the straight line BC shall be equal to the straight line EF.

Take* K, L, the centres of the circles, and join BK, * 1. 3.
KC, EL, LF; then, because the circumference BGC is equal to the circumference EHF, the angle BKC is equal* to the angle ELF: * 27. 3.
and because the circles ABC, DEF are equal, the straight lines from their centres are equal:* therefore BK, KC * 1 Def. 3.
are equal to EL, LF, and they contain equal angles: therefore the base BC is equal* to the base EF. Therefore, in equal circles, &c. * 4. 1.
Q. E. D.



PROP. XXX. PROB.

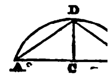
To bisect a given circumference, that is, to divide it into two equal parts.

Let ADB be the given circumference; it is required to bisect it.

Join AB, and bisect* it in C; from the point C * 10. 1.
draw* CD at right angles to AB: the circumference * 11. 1.
ADB shall be bisected in the point D.

Join AD, DB; and because AC is equal to CB, and CD common to the triangles ACD, BCD, the two sides

AC, CD are equal to the two BC, CD, each to ea and the angle ACD is equal to the angle BCD, because each of them is a right angle; therefore the base AD is



* 4. 1. equal* to the base BD: but equal

* 28. 3. straight lines cut off equal* circumferences, the grea equal to the greater, and the less to the less, and A DB are each of them less than a semicircle; beca

* Cor. 1. 3. DC passes through the centre:* wherefore the circu ference AD is equal to the circumference DB: the fore, the given circumference is bisected in D. Wh was to be done.

PROP. XXXI. THEOR.

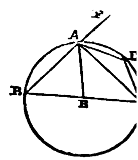
In a circle, the angle in a semicircle is a right ang the angle in a segment greater than a semicircle is than a right angle; and the angle in a segment than a semicircle is greater than a right angle.

Let ABCD be a circle, of which the diameter is] and centre E; and draw CA, dividing the circle i the segments ABC, ADC, and join BA, AD, DC; angle in the semicircle BAC shall be a right ang the angle in the segment ABC, which is greater tl a semicircle, shall be less than a right angle; and angle in the segment ADC, which is less than a sei circle, shall be greater than a right angle.

Join AE, and produce BA to F; and because] is equal to EA, the angle EAB is equal* to EB

* 5. 1.

also, because EA is equal to EC, the angle EAC is equal to ECA; wherefore the two angles ABC,



* 2 Ax. ACB are together equal* to the whole angle BAC: but FAC, the exterior angle of the triangle ABC,

1. is equal* to the two angles ABC,

ACB; therefore the angle BAC is equal* to the angle FAC, and therefore each of them is a right* angle: therefore the angle BAC in a semicircle is a right angle. * 1 Ax. * 10 Def 1.

And because the two angles ABC, BAC of the triangle ABC are together less* than two right angles, and that BAC has been proved to be a right angle, ABC must be less than a right angle; and therefore the angle in a segment ABC greater than a semicircle, is less than a right angle. * 17. 1.

And because ABCD is a quadrilateral figure in a circle, any two of its opposite angles are together equal* to two right angles; therefore the angles ABC, ADC are equal to two right angles; and ABC has been proved to be less than a right angle; therefore the other ADC is greater than a right angle. * 22. 3.

Besides, it is manifest that the circumference of the greater segment ABC falls without the right angle CAB, but the circumference of the less segment ADC falls within the right angle CAF. "And this is all that is meant, when in the Greek text, and the translations from it, the angle of the greater segment is said to be greater, and the angle of the less segment is said to be less, than a right angle." Q. E. D.

Cor. From this it is manifest, that if one angle of a triangle be equal to the other two, it is a right angle, because the angle adjacent to it is equal* to the same two; and when the adjacent angles are equal, they are right* angles. * 32. 1. * 10 Def. 1.

PROP. XXXII. THEOR.

If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles which this line makes with the line touching the circle, shall be equal to the angles which are in the alternate segments of the circle.

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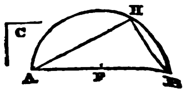
PROP. XXXIII. PROB.

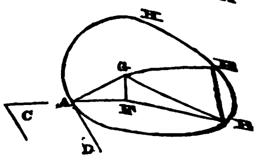
Upon a given straight line to describe a segment of a circle, which shall contain an angle equal to a given rectilineal angle.

Let AB be the given straight line, and C the given rectilineal angle; it is required to describe upon the given straight line AB, a segment of a circle, which shall contain an angle equal to the angle C.

First, let the angle C be a right angle; bisect* AB in F, and from the centre F, at the distance FB, describe the semicircle AHB; therefore the angle AHB in a semicircle is* equal to the right angle C. * 10

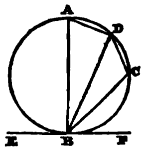


But, if the angle C be not a right angle, at the point A, in the straight line AB, make* the angle BAD equal to the angle C, and from the point A draw* AE at right angles to AD; bisect* AB in F, and from F draw* FG at right angles to AB, and join GB: and because AF is equal to FB, and FG common to the triangles AFG, BFG, the two sides AF FG, are equal to the two BF, FG, each to each; and the angle AFG is equal* to the angle BFG; therefore the base AG is equal* to the base GB; and therefore the circle described from the centre G, at the distance GA, shall pass through the point B; let this be the circle AHB: then the segment AHB shall contain an angle equal to the given angle C. Bec



Let the straight line EF touch the circle ABCD in B, and from the point B let the straight line BD be drawn, cutting the circle: the angles which BD makes with the touching line EF shall be equal to the angles in the alternate segments of the circle; that is, the angle DBF is equal to the angle which is in the segment DAB, and the angle DBE shall be equal to the angle in the segment DCB.

* 11. 1. From the point B draw* BA at right angles to EF, and take any point C in the circumference DB, and join AD, DC, CB; and because the straight line EF touches the circle ABCD in the point B, and BA is drawn at right angles to the touching line from the point of contact B, the



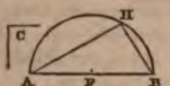
* 19. 3. centre of the circle is* in BA; therefore the angle ADB in a semicircle
* 31. 3. is a right* angle: and consequently the other two angles BAD, ABD are
* 32. 1. equal* to a right angle: but ABF
* Const. is likewise a right* angle; therefore the angle ABF is
* 1 Ax. equal* to the angles BAD, ABD: take from these equals the common angle ABD; therefore the remain-
* 3 Ax. ing angle DBF is equal* to the angle BAD, which is in the alternate segment of the circle; and because ABCD is a quadrilateral figure in a circle, the opposite angles
* 22. 3. BAD, BCD are equal* to two right angles; but the
* 13. 1. angles DBF, DBE are equal* to two right angles; therefore the angles DBF, DBE are equal to the angles
* 1 Ax. BAD, BCD;* and DBF has been proved equal to
* 2 Ax. BAD: therefore the remaining angle DBE is equal* to the angle BCD in the alternate segment of the circle. Wherefore, if a straight line, &c. Q. E. D.

PROP. XXXIII. PROB.

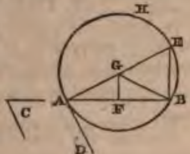
Upon a given straight line to describe a segment of a circle, which shall contain an angle equal to a given rectilineal angle.

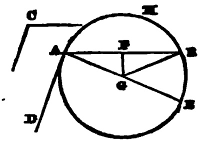
Let AB be the given straight line, and C the given rectilineal angle; it is required to describe upon the given straight line AB, a segment of a circle, which shall contain an angle equal to the angle C.

First, let the angle C be a right angle; bisect* AB in F, and from the centre F, at the distance FB, describe the semicircle AHB; therefore the angle AHB in a semicircle is* equal to the right angle C. * 10. 1. * 31. 3.



But, if the angle C be not a right angle, at the point A, in the straight line AB, make* the angle BAD equal to the angle C, and from the point A draw* AE at right angles to AD; bisect* AB in F, and from F draw* FG at right angles to AB, and join GB: and because AF is equal to FB, and FG common to the triangles AFG, BFG, the two sides AF FG, are equal to the two BF, FG, each to each; and the angle AFG is equal* to the angle BFG; therefore the base AG is equal* to the base GB; and therefore the circle described from the centre G, at the distance GA, shall pass through the point B; let this be the circle AHB: then the segment AHB shall contain *an angle equal* to the given angle C. Because * 23. 1. * 11. 1. * 10. 1. * 11. 1. * 10 Def. 1. * 4. 1.





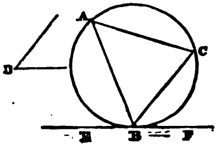
from the point A, the extremity of the diameter AE, AD is drawn at right angles to AE, (* Cor. 16. 3.) therefore AD* touches the circle; and because AB, drawn from the point of contact A, cuts the circle, the angle DAB (* 32. 3.) is equal to the angle in the alternate segment AHB:* (* Const.) but the angle DAB is equal* to the angle C; therefore the angle C is equal to the angle in the segment AHB: wherefore, upon the given straight line AB, the segment AHB of a circle is described, which contains an angle equal to the given angle C. Which was to be done.

PROP. XXXIV. PROB.

From a given circle to cut off a segment, which shall contain an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given rectilineal angle; it is required from the circle ABC to cut off a segment that shall contain an angle equal to the given angle D.

(* 17. 3.) Draw* the straight line EF, touching the circle ABC in the point B, and at the point B, in the straight line (* 23. 1.) BF, make* the angle FBC equal to the angle D: the segment BAC shall contain an angle equal to the given angle D.



Because the straight line EF touches the circle ABC, and BC is drawn from B, the point of contact, the (* 32. 3.) angle FBC is equal* to the angle in the alternate seg- (* Const.) ment BAC of the circle: but the angle FBC is equal* to

the angle D; therefore the angle in the segment BAC is equal* to the angle D: wherefore, from the given circle ABC, the segment BAC is cut off, containing an angle equal to the given angle D. Which was to be done. * 1 Ax.

PROP. XXXV. THEOR.

If two straight lines cut one another within a circle, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

Let the two straight lines AC, BD cut one another in the point E, within the circle ABCD; the rectangle contained by AE, EC shall be equal to the rectangle contained by BE, ED.

If each of the lines AC, BD pass through the centre, so that E is the centre; it is evident, that AE, EC, BE, ED are all equal, and the rectangle AE, EC is equal to the rectangle BE, ED.

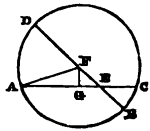
But let one of them BD pass through the centre, and cut at right angles the other AC, which does not pass through the centre, in the point E: then, if BD be bisected in F, F is the centre of the circle ABCD; join AF: and because BD, which passes through the centre, cuts at right angles in E, the straight line AC, which does not pass through the centre, AE is equal to EC. And because the straight line BD is cut into two equal parts in the point F, and into two unequal parts in the point E, the rectangle BE, ED, together with the square of EF, is equal* to the square of FB; that is, to the square of FA; but the squares of AE, EF are equal* to the square of AF; therefore the rectangle contained by BE, ED, *together* with the square of EF, is equal

* 5. 2.

* 47. 1.

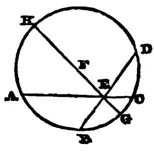
to the squares of AE, EF: take away the common square of EF, and the remaining rectangle BE, ED is
* 3 Ax. equal* to the remaining square of AE: that is, to the rectangle AE, EC; because AE was proved equal to EC.

Next, let BD which passes through the centre, cut the other AC, which does not pass through the centre, in E, but not at right angles: then, as before, if BD be bisected in F, F is the centre of the circle. Join
* 12. 1. AF, and from F draw* FG perpendicular to AC;
* 3. 3. therefore AG is equal* to GC; wherefore the rectangle AE, EC, together with the square of EG, is
* 5. 2. equal* to the square of AG: to each of these equals add the square of GF; therefore the rectangle AE,
* 2 Ax. EC, together with the squares of EG, GF, is equal* to the squares of AG, GF; but the
* 47. 1. squares of EG, GF are equal* to the square of EF; and the squares of AG, GF are equal to the square of AF: therefore the rectangle AE, EC, together with the square of EF, is equal to the square of AF; that is, to the square of



* 5. 2. FB: but the square of FB is equal* to the rectangle BE, ED, together with the square of EF; therefore the rectangle AE, EC, together with the square of EF,
* 1 Ax. is equal* to the rectangle BE, ED, together with the square of EF: take away the common square of EF,
* 3 Ax. and the remaining rectangle AE, EC is therefore equal* to the remaining rectangle BE, ED.

Lastly, let neither of the straight lines AC, BD pass
* 1. 3. through the centre. Take* the centre F, and through E, the intersection of the straight lines AC, DB, draw the diameter GEFH. Now the rectangle AE, EC has been proved equal to the rectangle GE, EH; and the rectangle BE, ED



equal to the same rectangle GE, EH; therefore the rectangle AE, EC is equal* to the rectangle BE, ED. *1 Ax.
Wherefore, if two straight lines, &c. Q. E. D.

PROP. XXXVI. THEOR.

If from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square of the line which touches it.

Let D be any point without the circle ABC, and DCA, DB be two straight lines drawn from it, of which DCA cuts the circle, and DB touches it: the rectangle AD, DC shall be equal to the square of DB.

Either DCA passes through the centre, or it does not; first, let it pass through E, the centre, and join EB; therefore the angle EBD is a right* angle: and because the straight line AC is bisected in E, and produced to the point D, the rectangle AD, DC, together with the square of EC, is equal* to the square of ED; but EC is equal to EB; therefore the rectangle AD, DC, together with the square of EB, is equal to the square of ED: but the square of ED is equal* to the squares of EB, BD, because EBD is a right angle: therefore the rectangle AD, DC, together with the square of EB, is equal* to the squares of EB, BD: take away the common square of EB; therefore the remaining rectangle AD, DC is equal* to the square of the tangent DB.

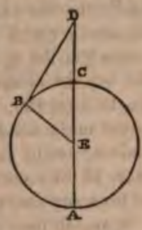
*18. 3.

*6. 2.

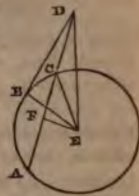
*47. 1.

*1 Ax.

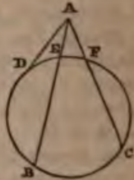
*3 Ax.



But if DCA does not pass through the centre of the circle ABC, take* E, the centre, and draw EF per- *1. 3.



* 12. 1. pendicular* to AC, and join EB, EC, ED: and because the straight line EF, which passes through the centre, cuts at right angles the straight line AC, which does not pass through the centre, it also bisects* it; therefore
* 3. 3.
AF is equal to FC: and because the straight line AC is bisected in F, and produced to D, the rectangle AD,
* 6. 2. DC, together with the square of FC, is equal* to the square of FD: to each of these equals add the square of FE; therefore the rectangle AD, DC, together with
* 2 Ax. the squares of CF, FE, is equal* to the squares of DF, FE: but because EFD is a right angle, the square of
* 47. 1. ED is equal* to the squares of DF, FE; and the square of EC to the squares of CF, FE; therefore the rectangle AD, DC, together with the square of EC, is
* 1 Ax. equal* to the square of ED: but CE is equal to EB; therefore the rectangle AD, DC, together with the square of EB, is equal to the square of ED: but because EBD is a right angle, the squares of EB, BD
* 47. 1. are equal* to the square of ED; therefore the rectangle AD, DC, together with the square of EB, is equal to the squares of EB, BD: take away the common square of EB; therefore the remaining rectangle
* 3 Ax. AD, DC is equal* to the square of DB. Wherefore, if from any point, &c. Q. E. D.



Cor. If from any point without a circle, there be drawn two straight lines cutting it, as AB, AC, the rectangles contained by the whole lines, and the parts of them without the circle, are equal to one another, viz. the rectangle BA, AE to the rectangle CA, AF: for each of them is equal to the square of the straight line AD, which touches the circle.

PROP. XXXVII. THEOR.

If from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square of the line which meets it, the line which meets shall touch the circle.

Let any point D be taken without the circle ABC, and from it let two straight lines DCA and DB be drawn, of which DCA cuts the circle, and DB meets it; if the rectangle AD, DC be equal to the square of DB, then DB shall touch the circle.

Draw* the straight line DE, touching the circle * 17. 3.
ABC, find F* its centre, and join FE, FB, FD; then * 1. 3.
FED is a right* angle: and because DE touches the * 18. 3.
circle ABC, and DCA cuts it, the rectangle AD, DC
is equal* to the square of DE: but the rectangle AD, * 36. 3.
DC is, by hypothesis, equal to the square of DB;
therefore the square of DE is equal* to the square of * 1 Ax.
DB; and the straight line DE equal to the straight
line DB: and FE is equal* to FB; wherefore DE, EF * 115 Def.
are equal to DB, BF, each to each; and the base FD is common to the two triangles DEF, DBF; therefore the angle DEF is equal* to the angle DBF; but DEF being a right angle, DBF also is a right angle: and BF, if produced, is a diameter, and the straight line which is drawn at right angles to a diameter, from the * 8. 1.

extremity of it, touches* the circle: therefore DB * Cor.16. 3.
touches the circle ABC. Wherefore, if from a point, &c. Q. E. D.

Cor. Hence it is evident, that if from a point without a circle, two straight lines be drawn to touch the circle, these two straight lines are equal to one another.

THE

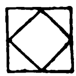
ELEMENTS OF EUCLID.

BOOK IV.

DEFINITIONS.

I.

A RECTILINEAL figure is said to be *inscribed* in another rectilineal figure, when all the angles of the inscribed figure are upon the sides of the figure in which it is inscribed, each upon each.

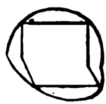


II.

In like manner, a figure is said to be *described* about another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.

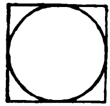
III.

A rectilineal figure is said to be *inscribed in a circle*, when all the angles of the inscribed figure are upon the circumference of the circle.



IV.

A rectilineal figure is said to be *described about a circle,* when each side of the circumscribed figure touches the circumference of the circle.

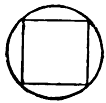


V.

In like manner, a circle is said to be *inscribed in a rectilineal figure,* when the circumference of the circle touches each side of the figure.

VI.

A circle is said to be *described about a rectilineal figure,* when the circumference of the circle passes through all the angular points of the figure about which it is described.



VII.

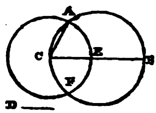
A straight line is said to be *placed in a circle,* when the extremities of it are in the circumference of the circle.

PROP. I. PROB.

In a given circle to place a straight line, equal to a given straight line, which is not greater than the diameter of the circle.

Let ABC be the given circle, and D the given straight line, not greater than the diameter of the circle: *it is required to* place in the circle ABC, a *straight line equal to D.*

Draw[a] CB the diameter of the circle ABC; then, if CB is equal to D, the thing required is done; for in the circle ABC a straight line CB is placed equal to D: but, if it is not, CB is greater than
* 3. 1. D: make CE equal* to D; and from the centre C, at the distance CE, describe the circle AEF, and join CA: CA shall be equal to D. Because C is the centre of the cir-



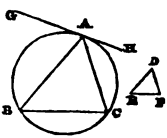
* Const. cle AEF, CA is equal to CE; but CE is equal* to D;
* 1 Ax. therefore CA is equal* to D: wherefore, in the circle ABC, a straight line CA is placed equal to the given straight line D, which is not greater than the diameter of the circle. Which was to be done.

PROP. II. PROB.

In a given circle to inscribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to inscribe in the circle ABC, a triangle equiangular to the triangle DEF.

* 7. 3. Draw* the straight line GAH touching the circle in the point A, and at the point A, in the straight line
* 23. 1. AH, make* the angle HAC equal to the angle DEF; and at the point A, in the straight line AG, make the angle GAB equal to the angle DFE, and join BC: ABC shall be the triangle required.



Because HAG touches the

[a] *Find the centre of the circle, and through it draw any straight line CB, terminated both ways by the circumference; this line is a diameter*

circle ABC, and AC is drawn from the point of contact, the angle HAC is equal* to the angle ABC in the alternate segment of the circle: but HAC is equal* to the angle DEF; therefore also the angle ABC is equal to DEF: for the same reason, the angle ACB is equal to the angle DFE; therefore the remaining angle BAC is equal* to the remaining angle EDF: wherefore the triangle ABC is equiangular to the triangle DEF, and it is inscribed in the circle ABC. Which was to be done.

* 32. 3.
* Const.
* 32. 1.

PROP. III. PROB.

About a given circle to describe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to describe a triangle about the circle ABC, equiangular to the triangle DEF.

Produce EF both ways to the points G, H; find* the centre K of the circle ABC, and from it draw any straight line KB; at the point K, in the straight line KB, make* the angle BKA equal to the angle DEG, and the angle BKC equal to the angle DFH; and through the points A, B, C, draw the straight lines LAM, MBN, NCL, touching* the circle ABC:[a] LMN shall be the triangle required.

* 1. 3.
* 23. 1.
* 17. 3.

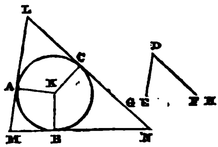
Because LM, MN, NL touch the circle ABC in the points A, B, C, to which from the centre are drawn KA, KB, KC, the angles at the points A, B, C, are right* angles: and because the four angles of the quadrilateral figure AMBK are equal to four right angles, for it can be divided into two triangles: and that two of them

* 18. 3

[a] It may be proved that MA, NC will meet, by joining AC; or since the angles at A, C are right angles, the angles CAL, ACL are together less than two right angles; therefore* MA, NC *will meet if produced.*

* 12 Ax. 1.

KAM, KBM are right angles, therefore the other two AKB, AMB are also equal to two right angles: but
* 13. 1. the angles DEG, DEF are likewise equal* to two right angles; therefore the angles AKB, AMB are equal to the angles DEG, DEF, of which AKB is equal to DEG; wherefore the remaining angle AMB is equal to the remaining angle DEF. In like manner, the angle LNM may be demonstrated to be equal to DFE; and therefore



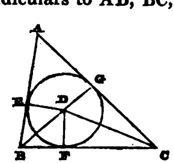
* 32. 1. the remaining angle MLN is equal* to the remaining angle EDF: wherefore the triangle LMN is equiangular to the triangle DEF: and it is described about the circle ABC. Which was to be done.

PROP. IV. PROB.

To inscribe a circle in a given triangle.

Let the given triangle be ABC: it is required to inscribe a circle in ABC.

* 9. 1. Bisect* the angles ABC, BCA by the straight lines BD, CD meeting one another in the point D, from
* 12. 1. which draw* DE, DF, DG perpendiculars to AB, BC, CA: and because the angle EBD is equal to the angle FBD, for the angle ABC is bisected by BD, and that the right angle BED is equal to the right angle BFD; therefore the two triangles EBD, FBD have two angles of the one equal to two angles of the other, each to each; and the side BD, which is opposite to one of the equal angles in each, is common to both; therefore their other sides



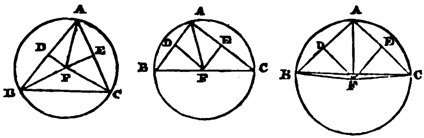
are equal;* wherefore DE is equal to DF: for the *26. 1.
same reason, DG is equal to DF; therefore DE is equal to DG, and the three straight lines DE, DF, DG are equal to one another; therefore the circle described from the centre D, at the distance of any of them, shall pass through the extremities of the other two, and touch the straight lines AB, BC, CA, because the angles at the points E, F, G, are right angles, and the straight line which is drawn from the extremity of a diameter at right angles to it, touches* the circle: *16. 3.
therefore each of the straight lines AB, BC, CA touches the circle, and the circle EFG is therefore inscribed in the triangle ABC. Which was to be done.

PROP. V. PROB.

To describe a circle about a given triangle.

Let the given triangle be ABC; it is required to describe a circle about ABC.

Bisect* AB, AC in the points D, E, and from these *10. 1
points draw DF, EF at right angles* to AB, AC; *11. 1.
DF, EF will meet if produced; for, if they do not



meet, they are parallel, wherefore AB, AC, which are at right angles to them, are parallel; which is absurd: let them meet in F, and join FA; also, if the point F be not in BC, join BF, CF: then, because AD is equal to DB, and DF common, and at right angles to AB, the base FA is equal* to the base FB: in like man- *4. 1.
ner, it may be proved that FC is equal to FA; and

therefore FB is equal to FC; and FA, FB, FC are equal to one another; wherefore the circle described from the centre F, at the distance of one of them, will pass through the extremities of the other two, and be described about the triangle ABC. Which was to be done.

COR. And it is manifest, that when the centre of the circle falls within the triangle, each of its angles
* 31. 3. is less than a right* angle, each of them being in a segment greater than a semicircle; but, when the centre is in one of the sides of the triangle, the angle opposite
* 31. 3. to this side, being in a semicircle, is a right* angle; and, if the centre falls without the triangle, the angle opposite to the side beyond which it is, being in a seg-
* 31. 3. ment less than a semicircle, is greater* than a right angle: therefore, if the given triangle be acute-angled, the centre of the circle falls within it; if it be a right-angled triangle, the centre is in the side opposite to the right angle; and, if it be an obtuse-angled triangle, the centre falls without the triangle, beyond the side opposite to the obtuse angle.

PROP. VI. PROB.

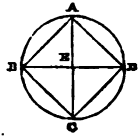
To inscribe a square in a given circle.

Let ABCD be the given circle; it is required to inscribe a square in ABCD.

Draw the diameters AC, BD at right angles to one another; and join AB, BC, CD, DA; the figure ABCD shall be the square required.

Because BE is equal to ED, for E is the centre, and that EA is common, and at right angles to BD; the
* 4. 1. base AB is equal* to the base AD; and, for the same reason, BC, CD are each of them equal to AB or AD; therefore the quadrilateral figure ABCD is equilateral. It is also rectangular; for the straight line BD, bein

the diameter of the circle ABCD, BAD is a semicircle: wherefore the angle BAD is a right* angle; for the same reason each of the angles ABC, BCD, CDA is a right angle; therefore the quadrilateral figure ABCD is rectangular, and it has been proved to be equilateral; therefore it is a* square; and it is inscribed in the circle ABCD. Which was to be done. * 31. 3.



* 30 Def. 1.

PROP. VII. PROB.

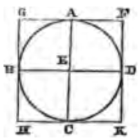
To describe a square about a given circle.

Let ABCD be the given circle; it is required to describe a square about ABCD.

Draw two diameters AC, BD of the circle ABCD, at right angles to one another, and through the points A, B, C, D, draw* FG, GH, HK, KF touching the circle; the figure GHKF shall be the square required. Because FG touches the circle ABCD, and EA is drawn from the centre E to A, the point of contact, the angles at A are right* angles: for the same reason, the angles at the points B, C, D, are right angles; and because the angles AEB and EBG are right angles, GH is parallel* to AC: for the same reason, AC is parallel to FK; and in like manner GF, HK may each of them be demonstrated to be parallel to BED; therefore the figures GK, GC, AK, FB, BK are parallelograms; and therefore GF is equal* to HK, and GH to FK; and because AC is equal to BD, and that AC is equal to each of the two GH, FK; and BD to each of the two GF, HK: GH, FK are each of *them equal to GF or HK*; therefore the quadrilateral

* 17. 3.

* 18. 3.



* 28. 1.

* 34. 1.

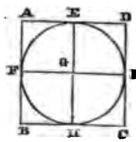
figure FGHK is equilateral. It is also rectangular; for GBEA being a parallelogram, and AEB a right
* 34. 1. angle, AGB* is likewise a right angle: and in the same manner it may be proved, that the angles at H, K, F, are right angles; therefore the quadrilateral figure FGHK is rectangular, and it was demonstrated
* 30 Def. 1. to be equilateral: it is therefore a* square; and it is described about the circle ABCD. Which was to be done.

PROP. VIII. PROB.

To inscribe a circle in a given square.

Let ABCD be the given square; it is required to inscribe a circle in ABCD.

* 10. 1. Bisect* each of the sides AB, AD, in the points F, E,
* 31. 1. and through E draw* EH parallel to AB or DC, and through F draw FK parallel to AD or BC; therefore each of the figures AK, KB, AH, HD, AG, GC, BG, GD, is a right-angled parallelogram or rectangle, and
* 34. 1. their opposite sides are equal;* and because AD is equal to AB, and that AE is the half of AD, and AF
* 7 Ax. the half of AB, AE is equal* to AF; wherefore the sides opposite to these are equal, viz. GF to GE; in the same manner, it may be demonstrated that GH, GK are each of them equal to GF or GE; therefore the four straight lines GE, GF, GH, GK, are equal to one another; and the circle described



from the centre G, at the distance of one of them, will pass through the extremities of the other three, and touch the straight lines AB, BC, CD, DA; because
* 29. 1. the angles at the points E, F, H, K are right* angles, and that the straight line which is drawn from the ex-

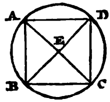
tremity of a diameter, at right angles to it, touches* the circle; therefore each of the straight lines AB, BC, CD, DA touches the circle, which therefore is inscribed in the square ABCD. Which was to be done. * Cor. 16. 3.

PROP. IX. PROB.

To describe a circle about a given square.

Let ABCD be the given square: it is required to describe a circle about ABCD.

Join AC, BD, cutting one another in E; and because DA is equal to AB, and AC common to the triangles DAC, BAC, the two sides DA, AC are equal to the two BA, AC, each to each; and the base DC is equal to the base BC; wherefore the angle DAC is equal* to the angle BAC, and the angle DAB is bisected by the straight line AC: in the same manner it may be demonstrated, that the angles ABC, BCD, CDA, are severally bisected by the straight lines BD, AC; therefore, because the angle DAB is equal to the angle ABC, and that the angle EAB is the half of DAB, and EBA the half of ABC; the angle EAB is equal* to the angle EBA; wherefore the side EA is equal* to the side EB. In the same manner it may be demonstrated, that the straight lines EC, ED are each of them equal to EA or EB; therefore the four straight lines EA, EB, EC, ED are equal to one another; and the circle described from the centre E, at the distance of one of them, will pass through the extremities of the other three, and be described about the square ABCD. Which was to be done.



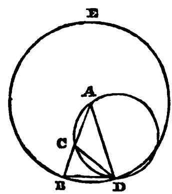
* 8. 1.
* 7 Ax.
* 6. 1.

PROP. X. PROB.

To describe an isosceles triangle, having each of the angles at the base double of the third angle.

* 11. 2. Take any straight line AB, and divide* it in the point C, so that the rectangle contained by AB, BC may be equal to the square of CA; and from the centre A, at the distance AB, describe the circle BDE, in
* 1. 4. which place* the straight line BD equal to AC, which is not greater than the diameter of the circle BDE; join DA, and the triangle ABD shall be such as is required, that is, each of the angles ABD, ADB is double of the angle BAD.

* 5. 4. Join DC, and about the triangle ADC describe* the circle ACD; then, because the rectangle AB, BC is
* Const. equal* to the square of AC, and that AC is equal to BD,
* 1 Ax. the rectangle AB, BC is equal* to the square of BD; and because from the point B, without the circle ACD, two straight lines BCA, BD are drawn to the circumference, one of which cuts, and the other meets the circle, and that the rectangle AB, BC, contained by the whole of the cutting line, and the part of it without the circle, is equal to the



square of BD which meets it: therefore the straight
* 37. 3. line BD touches* the circle ACD. And because BD touches the circle, and DC is drawn from D, the point
* 32 3. of contact, the angle BDC is equal* to the angle DAC in the alternate segment of the circle; to each of these equals add the angle CDA; therefore the whole angle
* 2 Ax. BDA is equal* to the two angles CDA, DAC; but the
* 32. 1. exterior angle BCD is equal* to the angles CDA,
* 1 Ax. DAC; therefore also BDA is equal* to BCD: but BDA

is equal* to the angle CBD, because the side AD is equal to the side AB; therefore CBD, or DBA is equal* to BCD; and consequently the three angles BDA, DBA, BCD, are equal to one another; and because the angle DBC is equal to the angle BCD, the side BD is equal* to the side DC; but BD was made equal to AC; therefore also CA is equal* to CD, and the angle CDA equal* to the angle CAD; therefore the angles CDA, CAD together, are double of the angle DAC: but BCD is equal* to the angles CDA, DAC; therefore also BCD is double of DAC, and BCD was proved to be equal to each of the angles BDA, DBA; therefore each of the angles BDA, DBA is double of the angle DAB; wherefore an isosceles triangle ABD is described, having each of the angles at the base double of the third angle. Which was to be done.

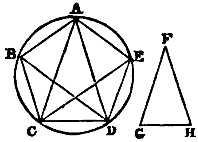
* 5. 1.
* 1 Ax.
* 6. 1.
* 1 Ax.
* 5. 1.
* 32. 1.

PROP. XI. PROB.

To inscribe an equilateral and equiangular pentagon in a given circle.

Let ABCDE be the given circle; it is required to inscribe an equilateral and equiangular pentagon in the circle ABCDE.

Describe* an isosceles triangle FGH, having each of the angles at G, H, double of the angle at F; and in the circle ABCDE inscribe* the triangle ACD equiangular to the triangle FGH, so that the angle CAD may be equal to the angle at F, and each of the angles ACD, CDA equal to the angle at G or H; wherefore each of the angles ACD, CDA is double of the angle CAD. Bisect* the angles ACD, CDA by the straight lines CE, DB; and join AB, BC, DE, EA. Then ABCDE shall be the *pentagon* required.



* 10. 4.
* 2. 4.
* 9. 1

Because each of the angles ACD, CDA is d
CAD, and that they are bisected by the straiş
CE, DB, therefore the five angles DAC, ACI
CDB, BDA are equal to one another; bı
* 26. 3. angles stand upon equal* circumferences; t
the five circumferences AB, BC, CD, DE,
equal to one another: and equal circumfereı
* 29. 3. subtended by equal* straight lines; therefore
straight lines AB, BC, CD, DE, EA are equa
another: therefore the pentagon ABCDE
lateral. It is also equiangular; for because
cumference AB is equal to the circumference
each of these equals add BCD; therefore tl
* 2 Ax. ABCD is equal* to the whole EDCB: but tl
AED stands on the circumference ABCD,
angle BAE on the circumference EDCB; t
* 27. 3. the angle BAE is equal* to the angle AED.
same reason, each of the angles ABC, BCD,
equal to the angle BAE, or AED: therefore t
tagon ABCDE is equiangular; and it has beeı
to be equilateral. Wherefore, in the given c
equilateral and equiangular pentagon has b
scribed. Which was to be done.

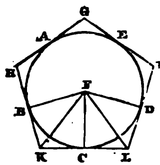
PROP. XII. PROB.

To describe an equilateral and equiangular p
about a given circle.

Let ABCDE be the given circle; it is reqı
describe an equilateral and equiangular pentagı
the circle ABCDE.

Let the angles of a pentagon, inscribed in th
by the last proposition, be in the points A, B, ı
so that the circumferences AB, BC, CD, DE,
* 11. 4. *equal;** and through the points A, B, C, D,
* 17. 3. *GH*, *HK*, KL, LM, MG, touching* th
the figure GHKLM shall be the pentagon v

Take the centre F, and join FB, FK, FC, FL, FD; and because the straight line KL touches the circle ABCDE in the point C, to which FC is drawn from F the centre, FC is perpendicular* to KL; therefore * 18. 3.
each of the angles at C is a right angle: for the same reason, the angles at the points B, D are right angles: and because FCK is a right angle, the square of FK is equal* to the squares of FC, CK: for the same * 47. 1.
reason, the square of FK is equal to the squares of FB, BK: therefore the squares of FC, CK are equal* to the * 1 Ax.
squares of FB, BK; but the square of FC is equal to the square of FB; therefore the remaining square of CK is equal* to the remaining square of BK, and the * 3 Ax.
straight line CK equal to BK: and because FB is equal to FC, and FK common to the triangles BFK, CFK, the two BF, FK are equal to the two CF, FK each to each; and the base BK was proved equal to the base KC; therefore the angle BFK is equal* to * 8. 1.
the angle KFC, and the angle BKF* to FKC; where- * 4. 1.
fore the angle BFC is double of the angle KFC, and BKC double of FKC: for the same reason, the angle CFD is double of the angle CFL, and CLD double of CLF: and because the circumference BC is equal to the circumference CD, the angle BFC is equal* to the angle CFD; and * 27. 3.
BFC is double of the angle KFC, and CFD double of CFL; therefore the angle KFC is equal* to * 7 Ax.
the angle CFL; and the right angle FCK is equal to the right angle FCL: therefore, in the two triangles FKC, FLC, there are two angles of the one equal to two angles of the other, each to each, and the side FC, which is adjacent to the equal angles in each, is common to both; therefore the other sides are equal* to * 26. 1.
the other sides, and the third angle to the third angle:



therefore the straight line KC is equal to CL, and the angle FKC to the angle FLC: and because KC is equal to CL, KL is double of KC. In the same manner it may be proved that HK is double of BK: and because BK is equal to KC, as was demonstrated, and that KL is double of KC, and HK double of BK,
* 6 Ax. therefore HK is equal* to KL: in like manner it may be proved that GH, GM, ML are each of them equal to HK or KL: therefore the pentagon GHKLM is equilateral. It is also equiangular; for, since the angle FKC is equal to the angle FLC, and that the angle HKL is double of the angle FKC, and KLM double of FLC, as was before demonstrated, the angle
* 6 Ax. HKL is equal* to KLM: and in like manner it may be proved that each of the angles KHG, HGM, GML is equal to the angle HKL or KLM: therefore the five angles GHK, HKL, KLM, LMG, MGH are equal to one another, and therefore the pentagon GHKLM is equiangular: and it was demonstrated to be equilateral: and it is described about the circle ABCDE. Which was to be done.

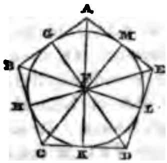
PROP. XIII. PROB.

To inscribe a circle in a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon; it is required to inscribe a circle in the pentagon ABCDE.

* 9. 1. Bisect* the angles BCD, CDE by the straight lines CF, DF, and from the point F, in which they meet, draw the straight lines FB, FA, FE: therefore since
* Hyp. BC is equal* to CD, and CF common to the triangles BCF, DCF, the two sides BC, CF are equal to the two
* Const. DC, CF, each to each; and the angle BCF is equal* to
* 4. 1. the angle DCF; therefore the base BF is equal* to

he base FD, and the other angles to the other angles, o which the equal sides are opposite; therefore the ıngle CBF is equal to the angle CDF: and because he angle CDE is double of CDF, and that CDE is equal to CBA, and CDF to CBF; CBA is also double of the angle CBF; therefore the angle ABF is equal to the angle CBF; wherefore the angle ABC is bisected by the straight line BF: in the same manner it may be demonstrated, that the angles BAE, AED, are bisected by the straight lines AF, FE. From the point F draw* FG, FH, FK, FL, FM perpendiculars to the straight lines AB, BC, CD, DE, EA: and because the angle HCF is equal to KCF, and the right angle FHC equal to the right angle FKC; therefore in the triangles FHC, FKC the two angles FHC, HCF are equal to the two FKC, KCF, each to each; and the side FC, which is opposite to one of the equal angles in each, is common to both; therefore the other sides are equal* each to each; wherefore the perpendicular FH is equal to the perpendicular FK. In the same manner it may be demonstrated that FL, FM, FG are each of them equal to FH or FK: therefore the five straight lines FG, FH, FK, FL, FM are equal to one another: wherefore the circle described from the centre F, at the distance of one of these five, will pass through the extremities of the other four, and touch the straight lines AB, BC, CD, DE, EA, because the angles at the points G, H, K, L, M are right angles; and that a straight line drawn from the extremity of the diameter of a circle at right angles to it, touches* the circle: therefore each of the straight lines AB, BC, CD, DE, EA *touches the circle; wherefore it is inscribed in the pentagon ABCDE.* Which was to be done.



* 12. 1.

* 26. 1.

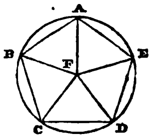
* 16. 3.

PROP. XIV. PROB.

To describe a circle about a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon; it is required to describe a circle about ABCDE.

* 9. 1. Bisect* the angles BCD, CDE by the straight lines CF, FD, and from the point F, in which they meet, draw the straight lines FB, FA, FE to the points B, A, E. It may be demonstrated, in the same manner as in the preceding proposition, that the angles CBA, BAE, AED are bisected by the straight lines FB, FA, FE: and because the angle BCD is equal to the angle CDE, and that FCD is the half of the angle BCD, and CDF the half of CDE; the angle FCD is



* 6. 1. equal to FDC: wherefore the side CF is equal* to the side FD. In like manner it may be demonstrated that FB, FA, FE are each of them equal to FC or FD: therefore the five straight lines FA, FB, FC, FD, FE are equal to one another; and the circle described from the centre F, at the distance of one of them, will pass through the extremities of the other four, and be described about the equilateral and equiangular pentagon ABCDE. Which was to be done.

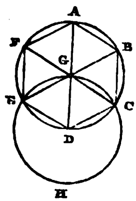
PROP. XV. PROB.

To inscribe an equilateral and equiangular hexagon in a given circle.

Let ABCDEF be the given circle; it is required to inscribe an equilateral and equiangular hexagon in the circle ABCDEF.

Find* G the centre of the circle ABCDEF, and draw * 1. 3.
the diameter AGD; and from D as a centre, at the distance DG, describe the circle EGCH, join EG, CG, and produce them to the points B, F; and join AB, BC, CD, DE, EF, FA: the hexagon ABCDEF shall be equilateral and equiangular.

Because G is the centre of the circle ABCDEF, GE is equal to GD: and because D is the centre of the circle EGCH, DE is equal to DG; wherefore GE is equal* to ED, and the triangle EGD is equilateral; and * 1 Ax.
therefore its three angles EGD, GDE, DEG are equal to one another;* but the three angles of a triangle are * Cor. 5. 1.
equal* to two right angles; therefore the angle EGD * 32. 1.
is the third part of two right angles: in the same manner it may be demonstrated that the angle DGC is also the third part of two right angles: and because the straight line GC makes with EB the adjacent angles EGC, CGB equal* to two right angles; the remaining angle CGB is * 13. 1.
the third part of two right angles; therefore the angles EGD, DGC, CGB, are equal to one another: and to these are equal* the vertical opposite angles BGA, AGF, FGE: therefore the six angles * 15. 1.
EGD, DGC, CGB, BGA, AGF, FGE are equal to one another: but equal angles stand upon equal* circumferences; therefore the six circumferences AB, BC, * 26. 3.
CD, DE, EF, FA are equal to one another; and equal circumferences are subtended by equal* straight lines; * 29. 3.
therefore the six straight lines AB, BC, CD, DE, EF, FA are equal to one another, and the hexagon ABCDEF is equilateral. It is also equiangular; for, since the circumference AF is equal to ED, to each of these equals add the circumference ABCD; therefore the whole circumference FABCD is equal to the whole



EDCBA: and the angle FED stands upon the circumference FABCD, and the angle AFE upon EDCBA;
* 27. 3. therefore the angle AFE is equal* to FED. In the same manner it may be demonstrated that the other angles of the hexagon ABCDEF are each of them equal to the angle AFE or FED: therefore the hexagon is equiangular; and it was proved to be equilateral; and it is inscribed in the given circle ABCDEF. Which was to be done.

Cor. From this it is manifest, that the side of the hexagon is equal to the straight line from the centre, that is, to the semidiameter of the circle.

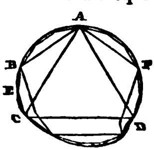
And if through the points A, B, C, D, E, F there be drawn straight lines touching the circle, an equilateral and equiangular hexagon will be described about it, which may be demonstrated from what has been said of the pentagon; and likewise a circle may be inscribed in a given equilateral and equiangular hexagon, and circumscribed about it, by a method like to that used for the pentagon.

PROP. XVI. PROB.

To inscribe an equilateral and equiangular quindecagon in a given circle.

Let ABCD be the given circle; it is required to inscribe an equilateral and equiangular quindecagon in the circle ABCD.

Let AC be the side of an equilateral triangle in-
* 2. 4. scribed* in the circle, and AB the side of an equilateral and equiangular pentagon
* 11. 4. inscribed* in the same; therefore, of such equal parts as the whole circumference ABCDF contains *fifteen*, the circumference ABC, *being* the third part of the whole,



contains five; and the circumference AB, which is the fifth part of the whole, contains three; therefore BC their difference contains two of the same parts: bisect* BC in E; therefore BE, EC are, each of them, the fifteenth part of the whole circumference ABCD: therefore, if the straight lines BE, EC be drawn, and straight lines equal to them be placed* round in the whole circle, an equilateral and equiangular quindecagon will be inscribed in it. Which was to be done.

* 30. 3.

* 1. 4.

And in the same manner as was done in the pentagon, if, through the points of division made by inscribing the quindecagon, straight lines be drawn touching the circle, an equilateral and equiangular quindecagon will be described about it: and likewise, as in the pentagon, a circle may be inscribed in a given equilateral and equiangular quindecagon, and circumscribed about it.[a]

[a] Besides the regular polygons treated of in this Book, and those which may be derived from them by successive bisections of the circumferences subtended by the sides of the polygons, there are others which may be described by elementary geometry, that is, by means of the straight line and circle. In 1801, the celebrated geometer, M. Gauss, of Göttingen, in a work entitled *Disquisitiones Arithmeticæ*, showed that every regular polygon, whose sides is a prime number of the form 2^n+1, may be inscribed in a circle, by elementary geometry. Hence a polygon of 17 sides may be inscribed in a circle, for 17 is a prime number, that is, a number which is not produced by the multiplication of any two whole numbers, and is of the prescribed form, viz., 2^4+1. So also may regular polygons of 257 and 65537 sides.

THE

ELEMENTS OF EUCLID.

BOOK V.

DEFINITIONS.

I.

A LESS magnitude is said to be a *part* of a greater magnitude, when the less measures the greater, that is, 'when the less is contained a certain number of 'times exactly in the greater.'

II.

A greater magnitude is said to be a *multiple* of a less, when the greater is measured by the less, that is, 'when the greater contains the less a certain number of times exactly.'

III.

"*Ratio* is a mutual relation of two magnitudes of the "same kind to one another, in respect of quantity."

IV.

Magnitudes are said to have a *ratio to one another*, when the less can be multiplied so as to exceed the other.

V.

The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth; or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth; or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

VI.

Magnitudes which have the same ratio are called *proportionals*. 'N. B. When four magnitudes are proportionals, it is usually expressed by saying, the 'first is to the second, as the third to the fourth.'

VII.

When of the equimultiples of four magnitudes (taken as in the fifth definition) the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth; then the first is said to have to the second a greater ratio than the third magnitude has to the fourth; and, on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

VIII.

"*Analogy*, or *proportion*, is the similitude or equality "of ratios."

IX.

Proportion consists in three terms at least.

X.

When three magnitudes are proportionals, the first is said to have to the third the ***duplicate*** ratio of that which it has to the second.

XI.

When four magnitudes are continual proportionals, the first is said to have to the fourth the ***triplicate*** ratio of that which it has to the second, and so on, ***quadruplicate***, &c. increasing the denomination still by unity, in any number of proportionals.

Definition A, to wit, of compound ratio.

When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

For example, if A, B, C, D be four magnitudes of the same kind, the first A is said to have to the last D the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D; or, the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D.

And if A has to B the same ratio which E has to F, and B to C the same ratio that G has to H; and C to D the same that K has to L: then, by this definition, A is said to have to D the ratio compounded of ratios which are the same with the ratios of E to F, G to H, and K to L. And the same thing is to be understood when it is more briefly expressed by saying, A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the same things being supposed, if M has to N the same ratio which A has to D; then, for shortness' sake, M is said to have to N the ratio

compounded of the ratios of E to F, G to H, and K to L.

XII.

In proportionals, the *antecedent* terms are called *homologous* to one another, as also the *consequents* to one another.

'Geometers make use of the following technical words 'to signify certain ways of changing either the order 'or magnitude of proportionals, so that they continue 'still to be proportionals.'

XIII.

Permutando, or *alternando*, by permutation, or alternately. This word is used when there are four proportionals, and it is inferred, that the first has the same ratio to the third, which the second has to the fourth; or that the first is to the third, as the second to the fourth; as is shown in the 16th Prop. of this fifth Book.

XIV.

Invertendo, by *inversion;* when there are four proportionals, and it is inferred, that the second is to the first, as the fourth to the third. Prop. B. Book 5.

XV.

Componendo, by *composition;* when there are four proportionals, and it is inferred, that the first together with the second, is to the second, as the third together with the fourth, is to the fourth. Prop. 18th, Book 5.

XVI.

Dividendo, by *division;* when there are four proportionals, and it is inferred, that the excess of the first above the second, is to the second, as the excess of the third *above the fourth*, is to the fourth. Prop. *17th, Book 5.*

XVII.

Convertendo, by *conversion;* when there are four proportionals, and it is inferred, that the first is to its excess above the second, as the third to its excess above the fourth. Prop. E. Book 5.

XVIII.

Ex æquali (sc. distantia), or *ex æquo*, from equality of distance; when there is any number of magnitudes more than two, and as many others, such that they are proportionals when taken two and two of each rank, and it is inferred, that the first is to the last of the first rank of magnitudes, as the first is to the last of the others: 'Of this there are the two following kinds, which arise from the different order in 'which the magnitudes are taken two and two.'

XIX.

Ex æquali, from equality; this term is used simply by itself, when the first magnitude is to the second of the first rank, as the first to the second of the other rank; and as the second is to the third of the first rank, so is the second to the third of the other; and so on in order, and the inference is as mentioned in the preceding definition; whence this is called ordinate proportion. It is demonstrated in Prop. 22nd. Book 5.

XX.

Ex æquali, in proportione perturbatâ, seu inordinatâ, from equality in perturbate or disorderly proportion; this term is used when the first magnitude is to the second of the first rank, as the last but one is to the last of the second rank; and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank, so is the third

from the last to the last but two of the second rank; and so on in a cross order: and the inference is as in the 18th definition. It is demonstrated in Prop. 23rd. of Book 5.

AXIOMS.

I.

EQUIMULTIPLES of the same, or of equal magnitudes, are equal to one another.[a]

II.

Those magnitudes, of which the same or equal magnitudes are equimultiples, are equal to one another.

III.

A multiple of a greater magnitude is greater than the same multiple of a less.

IV.

That magnitude, of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

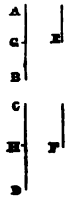
PROP. I. THEOR.

If any number of magnitudes be equimultiples of as many, each of each; what multiple soever any one of them is of its part, the same multiple shall all the first magnitudes be of all the other.

Let any number of magnitudes AB, CD be equimultiples of as many others E, F, each of each; what multiple soever AB is of E, the same multiple shall AB and CD together, be of E and F together.

[a] *Equimultiples of magnitudes* are multiples that contain them *respectively the same number of times.*

Because AB is the same multiple of E that CD is of F, as many magnitudes as there are in AB equal to E, so many are there in CD equal to F. Divide AB into magnitudes equal to E, viz. AG, GB; and CD into CH, HD equal each of them to F: therefore the number of the magnitudes CH, HD shall be equal to the number of the others AG, GB: and because AG is equal to E, and CH to F, therefore AG and CH together are equal to* E and F together: for the same reason, because GB is equal to E, and HD to F; GB and HD together are equal to E and F together: wherefore, as many magnitudes as are in AB equal to E, so many are there in AB, CD together, equal to E and F together. Therefore, what multiple soever AB is of E, the same multiple is AB and CD together, of E and F together.



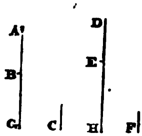
* 2 Ax. 1.

Therefore, if any magnitudes, how many soever, be equimultiples of as many, each of each; what multiple soever any one of them is of its part, the same multiple shall all the first magnitudes be of all the other; 'For 'the same demonstration holds in any number of mag-'nitudes, which was here applied to two.' Q. E. D.

PROP. II. THEOR.

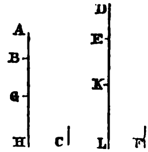
If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth; then shall the first together with the fifth be the same multiple of the second, that the third together with the sixth is of the fourth.

Let AB the first be the same multiple of C the second, that DE the third is of F the fourth; and BG the fifth the same multiple of C the second, that EH the sixth is of F the fourth: then shall AG, the first



together with the fifth, be the same multiple of C the second, that DH, the third together with the sixth, is of F the fourth.

Because AB is the same multiple of C, that DE is of F; there are as many magnitudes in AB equal to C, as there are in DE equal to F: in like manner, as many as there are in BG equal to C, so many are there in EH equal to F: therefore as many as there are in the whole AG equal to C, so many are there in the whole DH equal to F: therefore AG is the same multiple of C that DH is of F; that is, AG, the first and fifth together, is the same multiple of C the second, that DH, the third and sixth together, is of F the fourth. If, therefore, the first be the same multiple, &c. Q. E. D.



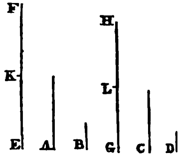
COR. From this it is plain, that if any number of magnitudes AB, BG, GH, be multiples of another C; and as many DE, EK, KL be the same multiples of F, each of each; then the whole of the first, viz. AH, is the same multiple of C, that the whole of the last, viz. DL, is of F.

PROP. III. THEOR.

If the first be the same multiple of the second, which the third is of the fourth; and if of the first and third there be taken equimultiples, these shall be equimultiples, the one of the second, and the other of the fourth.

Let A the first be the same multiple of B the second, that C the third is of D the fourth; and of A, C let the *equimultiples EF,* GH be taken: then EF shall *be the same multiple of B,* that GH is of D.

Because EF is the same multiple of A, that GH is of C, there are as many magnitudes in EF equal to A, as there are in GH equal to C: let EF be divided into the magnitudes EK, KF, each equal to A; and GH into GL, LH each equal to C: therefore the number of the magnitudes EK, KF, are equal to the number of the others GL, LH: and because A is the same multiple of B, that C is of D, and that EK is equal to A, and GL to C; therefore EK is the same multiple of B, that GL is of D: for the same reason, KF is the same multiple of B, that LH is of D; and so, if there be more parts in EF, GH equal to A, C: then because EK the first is the same multiple of B the second, which GL the third is of D the fourth, and that KF the fifth is the same multiple of B the second, which LH the sixth is of D the fourth; therefore EF, the



* 2. 5. first together with the fifth, is the same multiple* of B the second, which GH, the third together with the sixth, is of D the fourth. If, therefore, the first, &c. Q. E. D.

PROP. IV. THEOR.

If the first of four magnitudes has the same ratio to the second which the third has to the fourth; then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth, viz. 'the equimultiple of the first shall have the same ratio to that of the second, which the equimultiple of the third has to that of the fourth.'

Let A the first have to B the second, the same ratio which C the third has to D the fourth; and of A, C let there be taken any equimultiples whatever E, F;

and of B, D any equimultiples whatever G, H; then E shall have the same ratio to G, which F has to H.

Take of E, F any equimultiples whatever K, L, and of G, H, any equimultiples whatever M, N: then, because E is the same multiple of A, that F is of C; and of E and F have been taken equimultiples K, L; therefore K is the same multiple of A, that L is of C:* for the same reason, M is the same multiple of B, that N is of D: and because, as A is to B, so is C to D,* and of A and C have been taken certain equimultiples K and L; and of B and D have been taken certain equimultiples M and N; if therefore K be greater than M, L is greater than N: if equal, equal; and if less, less;* but K, L are any equimultiples whatever of E, F; and M, N any whatever of G, H: therefore as E is to G, so is* F to H. Therefore, if the first, &c. Q. E. D.

* 3. 5.
* Hyp.
* 5 Def. 5.
* 5 Def. 5.

COR. Likewise, if the first has the same ratio to the second, which the third has to the fourth, then also any equimultiples whatever of the first and third shall have the same ratio to the second and fourth: and in like manner, the first and the third shall have the same ratio to any equimultiples whatever of the second and fourth.

Let A the first have to B the second, the same ratio which C the third has to D the fourth, and of A and C let E and F be any equimultiples whatever; then E shall be to B, as F to D.

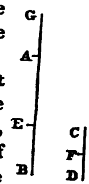
Take of E, F any equimultiples whatever K, L, and of B, D any equimultiples whatever G, H; then it may be demonstrated, as before, that K is the same multiple

* Hyp. of A, that L is of C: and because* A is to B, as C is to D, and of A and C certain equimultiples have been taken, viz. K and L; and of B and D certain equimultiples, viz. G, H; therefore, if K be greater than G, L is greater than H; if equal, equal; and if less,
* 5 Def. 5. less:* but K, L are any equimultiples whatever of E, F, and G, H any whatever of B, D; therefore, as E is
* 5 Def. 5. to B, so is F to D.* In the same way the other case is demonstrated.

PROP. V. THEOR.

If one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other; the remainder shall be the same multiple of the remainder, that the whole is of the whole.

Let the magnitude AB be the same multiple of CD, that AE taken from the first, is of CF taken from the other; the remainder EB shall be the same multiple of the remainder FD, that the whole AB is of the whole CD.



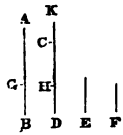
Take AG the same multiple of FD, that
* 1. 5. AE is of CF: therefore AE is* the same multiple of CF, that EG is of CD: but AE, by the hypothesis, is the same multiple of CF, that AB is of CD: therefore EG is the same multiple of CD that AB is of CD; wherefore EG
* 1 Ax. 5. is equal* to AB; from each of these equals take the common magnitude AE; and the remainder AG is equal to the remainder EB. Wherefore, since AE is
* Const. the same multiple of CF, that AG is of FD,* and that AG has been proved equal to EB; therefore AE is the same multiple of CF, that EB is of FD: but AE is
* Hyp. the same multiple of CF* that AB is of CD; therefore EB is the same multiple of FD, that AB is of CD.

Therefore, if one magnitude, &c. Q. E. D.

PROP. VI. THEOR.

If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two; the remainders are either equal to these others, or equimultiples of them.

Let the two magnitudes AB, CD be equimultiples of the two E, F, and let AG, CH taken from the first two be equimultiples of the same E, F; the remainders GB, HD shall be either equal to E, F, or equimultiples of them.

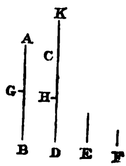


First, let GB be equal to E; HD shall be equal to F. Make CK equal to F; and because AG is the same multiple* of E, that CH is of F, and that GB is equal to E, and CK to F; therefore AB is the same multiple of E, that KH is of F: but AB, by the hypothesis, is the same multiple of E that CD is of F; therefore KH is the same multiple of F, that CD is of F; wherefore KH is equal* to CD: and from each of these equals take the common magnitude CH, then the remainder KC is equal to the remainder HD: but KC is equal to F;* therefore HD is equal to F.

* Hyp.

* 1 Ax. 5.

* Const.



Next let GB be a multiple of E; then HD shall be the same multiple of F. Make CK the same multiple of F, that GB is of E: and because AG is the same multiple* of E, that CH is of F; and GB the same multiple of E, that CK is of F: therefore AB is the same multiple of E, that KH is of F:* but AB is the same multiple* of E, that CD is of F; therefore KH is the same multiple of F, that CD is of F: wherefore KH is equal* to CD: and from each of these equals take

* Hyp.

* 2. 5.

* Hyp.

* 1 Ax. 5.

CH; therefore the remainder KC is equal to the remainder HD: and because GB is the same multiple of E,* that KC is of F, and that KC is equal to HD; therefore HD is the same multiple of F, that GB is of E. If, therefore, two magnitudes, &c. Q. E. D.

* Const.

PROP. A. THEOR.

If the first of four magnitudes has the same ratio to the second, which the third has to the fourth; then, if the first be greater than the second, the third is also greater than the fourth; if equal, equal; and if less, less.

Take any equimultiples of each of them, as the doubles of each; then, by Def. 5th of this Book, if the double of the first be greater than the double of the second, the double of the third is greater than the double of the fourth; but, if the first be greater than the second, the double of the first is greater than the double of the second; wherefore also the double of the third is greater than the double of the fourth; therefore the third is greater than the fourth: in like manner, if the first be equal to the second, or less than it, the third can be proved to be equal to the fourth, or less than it. Therefore, if the first, &c. Q. E. D.

PROP. B. THEOR.

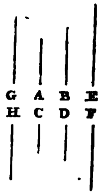
If four magnitudes are proportionals, they are proportionals also when taken inversely.

Let A be to B, as C is to D, then also inversely B shall be to A, as D to C.

Take of B and D any equimultiples whatever E and F; and of A and C any equimultiples whatever G and H. First, let E be greater than G, then G is less than E; and because A is to B, as C is to D,* and of A and

* Hyp

C, the first and third, G and H are equimultiples; and of B and D, the second and fourth, E and F are equimultiples; and that G is less than E, therefore H is* less than F; that is, F is greater than H; if, therefore, E be greater than G, F is greater than H: in like manner, if E be equal to G, F may be proved to be equal to H; and if less, less; but E and F are any equimultiples* whatever of B and D, and G and H any whatever of A and C; therefore,* as B is to A, so is D to C. If, then, four magnitudes, &c. Q. E. D.



* 5 Def. 5.

* Const.

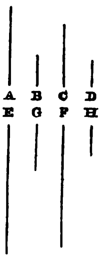
* 5 Def. 5.

PROP. C. THEOR.

If the first be the same multiple of the second; or the same part of it, that the third is of the fourth; the first is to the second, as the third is to the fourth.

Let A the first be the same multiple of B the second, that C the third is of D the fourth: A shall be to B as C is to D.

Take of A and C any equimultiples whatever E and F; and of B and D any equimultiples whatever G and H: then, because A is the same* multiple of B that C is of D; and that E is the same* multiple of A, that F is of C; therefore E is the same multiple of B, that F is of D;* that is, E and F are equimultiples of B and D; but G and H are equimultiples* of B and D; therefore, if E be a greater multiple of B than G is of B, F is a greater multiple of D than H is of D; that is, if E be greater than G, F is greater than H: in like manner, if E be equal to G, or less than it, F is equal to H, or less than it. But E, F are any equi-



* Hyp.

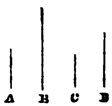
* Const.

* 3. 5.

* Const.

* Const. multiples* whatever, of A, C, and G, H any equimultiples whatever of B, D: therefore A is to B, as C is
* 5 Def. 5. to D.*

Next, let A the first be the same part of B the second, that C the third is of D the fourth: A shall be to B, as C is to D. For since A is the same part of B that C is of D; therefore B is the same multiple of A, that D is of C: wherefore, by the pre-



* B. 5. ceding case, B is to A, as D is to C; and inversely,* A is to B, as C is to D. Therefore, if the first be the same multiple, &c. Q. E. D.

PROP. D. THEOR.

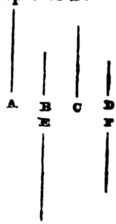
If the first be to the second as the third to the fourth, and if the first be a multiple, or part of the second; the third is the same multiple, or the same part of the fourth.

Let A be to B, as C is to D; and first let A be a multiple of B; C shall be the same multiple of D.

Take E equal to A, and whatever multiple A or E is of B, make F the

* Hyp. same multiple of D: then, because* A is to B, as C is to D; and of B the second, and D the fourth equimultiples E and F have been taken; therefore

* Cor. 4. 5. A is to E, as C to F:* but A is equal
* A. 5. to E, therefore C is equal to F:* but
* Const. F is the same* multiple of D, that A is of B: therefore C is the same multiple of D, that A is of B.



See the figure at the top of this page. Next, let A the first be a part of B the second; C the third shall be the same part of D the fourth.

Because A is to B, as C is to D; then, inversely, B
* B. 5. is* to A, as D to C: but A is a part of B, therefore B

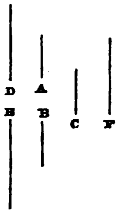
is a multiple of A; and, by the preceding case, D is the same multiple of C; that is, C is the same part of D, that A is of B. Therefore, if the first, &c. Q. E. D.

PROP. VII. THEOR.

Equal magnitudes have the same ratio to the same magnitude: and the same has the same ratio to equal magnitudes.

Let A and B be equal magnitudes, and C any other; A and B shall each of them have the same ratio to C, and C shall have the same ratio to each of the magnitudes A and B.

Take of A and B any equimultiples whatever D and E, and of C any multiple whatever F: then, because D is the same* multiple of A, that E is * Const.
of B, and that A is equal* to B; there- * Hyp.
fore D is* equal to E: therefore, if D * 1 Ax. 5.
be greater than F, E is greater than F; if equal, equal; and if less, less: but D and E are any equimultiples* of * Const.
A and B, and F is any multiple of C; therefore,* as A is to C, so is B to C. * 5 Def. 5.



Likewise C shall have the same ratio to A, that it has to B. For, having made the same construction, D may in like manner be proved to be equal to E: therefore, if F be greater than D, it is likewise greater than E; if equal, equal; and if less, less: but F is any multiple whatever of C, and D, E are any equimultiples whatever of A, B; therefore C is to A, as C is to B.* Therefore equal magnitudes, * 5 Def. 5.
&c. Q. E. D.

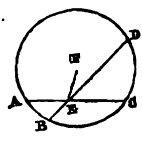
equal to two angles in the other, and the side EF, which is opposite to one of the equal angles in each, is common to both; therefore the other sides are
* 26. 1. *equal; and therefore AF is equal to FB. Wherefore, if a straight line, &c. Q. E. D.

PROP. IV. THEOR.

If in a circle two straight lines cut one another, which do not both pass through the centre, they do not bisect each other.

Let ABCD be a circle, and AC, BD two straight lines in it which cut one another in the point E, and do not both pass through the centre: AC, BD shall not bisect one another.

For, if it is possible, let AE be equal to EC, and BE to ED; then, if one of the lines pass through the centre, it is plain that it cannot be bisected by the other which does not pass through the centre: but if neither of them
* 1. 3. pass through the centre, take* F the centre of the circle, and join EF: and because FE, a straight line through the centre, bisects another

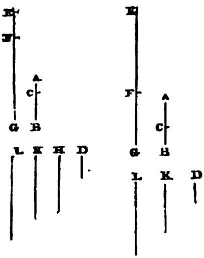


AC which does not pass through the centre, it cuts it
* 3. 3. at right* angles; wherefore FEA is a right angle. Again, because the straight line FE bisects the straight line BD, which does not pass through the centre, it
* 3. 3. cuts it at right* angles; wherefore FEB is a right angle: but FEA was proved to be a right angle; therefore FEA is equal to the angle FEB, the less to the greater, which is impossible: therefore AC, BD do not bisect one another. Wherefore, if in a circle, &c. *Q. E. D.*

tiple of CB, that EG is of AB:* wherefore EG and FG are equimultiples of AB and CB; and it was proved, that FG is not less than K, and, by the construction, EF is greater than D; therefore the whole EG is greater than K and D together: but K together with D is equal* to L; therefore EG is greater than L; but FG is not greater* than L; and EG, FG were proved to be equimultiples of AB, BC, and L is a multiple* of D; therefore* AB has to D a greater ratio than BC has to D.

* 1. 5.
* Const.
* Const.
* Const.
* 7 Def. 5.

Fig. 2. *Fig.* 3.



Also D shall have to BC a greater ratio than it has to AB. For having made the same construction, it may, in like manner, be proved that L is greater than FG, but that it is not greater than EG: and L is a multiple* of D; and FG, EG were proved to be equimultiples of CB, AB; therefore D has to CB a greater ratio* than it has to AB. Wherefore, of unequal magnitudes, &c. Q. E. D.

* Const.
* 7 Def. 5

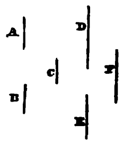
PROP. IX. THEOR.

Magnitudes which have the same ratio to the same magnitude are equal to one another: and those to which the same magnitude has the same ratio are equal to one another.

Let A and B have each of them the same ratio to C: A shall be equal to B.

For, if they are not equal, one of them must be greater than the other; let A be the greater; then, by what was shown in the preceding proposition, there are some equimultiples of A and B, and some multiple

of C such, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than that of C. Let these multiples be taken, and let D, E, be the equimultiples of A, B, and F the multiple of C, such that D may be greater than F, but E not
* Hyp. greater than F: then, because* A is to C, as B is to C, and of A, B, are taken equimultiples D, E, and of C is taken a multiple F; and that D is greater than F; therefore
* 5 Def. 5. E is also greater* than F: but this is impossible, because E is not greater than F, by construction; therefore A and B are not unequal; that is, they are equal.



Next, let C have the same ratio to each of the magnitudes A and B; A shall be equal to B.

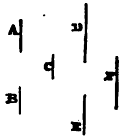
For, if they are not equal, one of them must be greater than the other; let A be the greater; then, as was proved in Prop. 8th, there is some multiple F of C; and some equimultiples E and D of B and A such, that F is greater than E, but not greater than D; and
* Hyp. because* C is to B, as C is to A, and that F the multiple of the first, is greater than E the multiple of the second; therefore F the multiple of the third, is greater
* 5 Def 5. than D the multiple of the fourth:* but this is impossible, because F is not greater than D. Therefore, A is equal to B. Wherefore, magnitudes which, &c. Q. E. D.

PROP. X. THEOR.

That magnitude which has a greater ratio than another has unto the same magnitude, is the greater of the two: and that magnitude to which the same has a greater ratio than it has unto another magnitude, is the lesser of the two.

Let A have to C a greater ratio than B has to C; A shall be greater than B.

For, because A has to C a greater ratio than B has to C, there are* some equimultiples of A and B, and some multiple of C such, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than it: let them be taken, and let D, E be the equimultiples of A, B, and F the multiple of C such, that D is greater than F, but E is not greater than F: therefore D is greater than E: and because D and E are equimultiples of A and B, and that D is greater than E; therefore A is* greater than B. * 7 Def. 5.



* 4 Ax. 5.

Next, let C have to B a greater ratio than C has to A; B shall be less than A.

For* there is some multiple F of C, and some equimultiples E and D of B and A such, that F is greater than E, but not greater than D; therefore E is less than D; and because E and D are equimultiples of B and A, therefore B is* less than A. Therefore, that magnitude, &c. Q. E. D. * 7 Def. 5. * 4 Ax. 5.

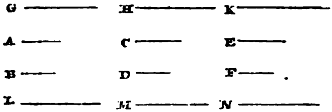
PROP. XI. THEOR.

Ratios that are the same to the same ratio, are the same to one another.

Let A be to B as C is to D; and let C be to D as E is to F; A shall be to B, as E to F.

Take of A, C, E, any equimultiples whatever G, H, K; and of B, D, F, any equimultiples whatever L, M, N. And since A is to B, as C to D, and that G, H are taken equimultiples of A, C, and L, M of B, D; therefore if G be greater than L, H is greater than M; and if equal, equal; and if less, less.* Again, because C is to D, as E is to F, and that H, K are taken equimultiples of C, E; and M, N, of D, F; therefore if H be greater than M, K is greater than N; and if equal, * 5 Def. 5.

equal; and if less, less; but it has been proved that if G be greater than L, H is greater than M: and if



equal, equal; and if less, less; therefore, if G be greater than L, K is greater than N; and if equal, equal; and if less, less: but G, K are any equimultiples whatever of A, E; and L, N any whatever of B, F: therefore, as A is to B, so is E to F.* Wherefore, ratios that, &c. Q. E. D.

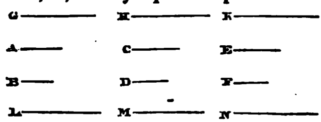
* 5 Def. 5.

PROP. XII. THEOR.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.

Let any number of magnitudes A, B, C, D, E, F, be proportionals; that is, as A is to B, so C to D, and E to F: as A is to B, so shall A, C, E together, be to B, D, F together.

Take of A, C, E any equimultiples whatever G, H,



K; and of B, D, F any equimultiples whatever L, M, N: then because A is to B, as C is to D, and as E to F; and that G, H, K are equimultiples of A, C, E, and L, M, N equimultiples of B, D, F; therefore if G be greater than L, H is greater than M, and K greater than N; and if equal, equal; and if less, less:* therefore if G be greater than L, then G, H, K together,

* 5 Def. 5.

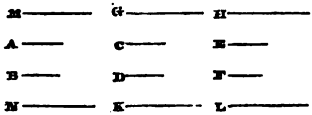
are greater than L, M, N together; and if equal, equal; and if less, less. But G, and G, H, K together, are any equimultiples of A, and A, C, E together; because, if there be any number of magnitudes equimultiples of as many, each of each, whatever multiple one of them is of its part, the same multiple is the whole of the whole:* for the same reason L, and L, M, N are any * 1. 5.
equimultiples of B, and B, D, F: therefore as A is to B,* so are A, C, E together to B, D, F together. * 5 Def. 5.
Wherefore, if any number, &c. Q. E. D.

PROP. XIII. THEOR.

If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth; the first shall also have to the second a greater ratio than the fifth has to the sixth.

Let A the first have the same ratio to B the second, which C the third has to D the fourth, but C the third to D the fourth, a greater ratio than E the fifth has to F the sixth: A the first shall have to B the second, a greater ratio than E the fifth has to F the sixth.

Because C has to D a greater ratio than E has to F, there are some equimultiples of C and E, and some of D and F such, that the multiple of C is greater than the multiple of D, but the multiple of E not greater



than the multiple of F:* let such be taken, and let * 7 Def. 5.
G, H be equimultiples of C, E and K, L equimultiples of D, F, such that G may be greater than K, but H not greater than L; and whatever multiple G is of C, take

M the same multiple of A; and whatever multiple K is of D, take N the same multiple of B: then, because
* Hyp. A is to B,* as C to D, and of A and C, M and G are equimultiples: and of B and D, N and K are equimultiples; therefore if M be greater than N, G is
* 5 Def. 5. greater than K; and if equal, equal; and if less, less;*
* Const. but G is greater* than K, therefore M is greater than
* Const. N: but H is not greater* than L; and M H are equimultiples of A, E; and N, L equimultiples of B, F:
* 7 Def. 5. therefore A has to B a greater ratio than E has to F.* Wherefore, if the first, &c. Q. E. D.

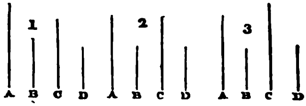
COR. And if the first have to the second a greater ratio than the third has to the fourth, but the third to the fourth the same ratio which the fifth has to the sixth; it may be demonstrated, in like manner, that the first has to the second a greater ratio than the fifth has to the sixth.

PROP. XIV. THEOR.

If the first have to the second the same ratio which the third has to the fourth; then, if the first be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, less.

Let A the first have to B the second, the same ratio which C the third has to D the fourth; if A be greater than C, B shall be greater than D.

Because A is greater than C, and B is any other magnitude, A has to B a greater ratio than C has to



* 8. 5. B:* but, as A is to B, so is C to D; therefore also C
* 13. 5. has to D a greater ratio than C has to B:* but of two

magnitudes, that to which the same has the greater ratio is the lesser of the two:* therefore D is less than B; that is, B is greater than D. * 10. 5.

Secondly, if A be equal to C, B shall be equal to D. For A is to B, as C, that is, A to D; therefore B is equal to D.* * 9. 5.

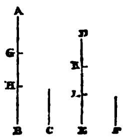
Thirdly, if A be less than C, B shall be less than D. For C is greater than A, and because C is to D, as A is to B, therefore D is greater than B, by the first case; wherefore B is less than D. Therefore, if the first, &c. Q. E. D.

PROP. XV. THEOR.

Magnitudes have the same ratio to one another which their equimultiples have.

Let AB be the same multiple of C, that DE is of F: C shall be to F, as AB is to DE.

Because AB is the same multiple of C, that DE is of F; there are as many magnitudes in AB equal to C, as there are in DE equal to F. Let AB be divided into magnitudes, each equal to C, viz. AG, GH, HB; and DE into magnitudes, each equal to F, viz. DK, KL, LE:



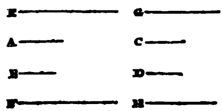
then the number of the first AG, GH, HB, is equal to the number of the last DK, KL, LE: and because AG, GH, HB are all equal, and that DK, KL, LE are also equal to one another; therefore AG is to DK, as GH to KL, and as HB to LE:* but as one of the antecedents is to its consequent, so are all the antecedents together to all the consequents together;* wherefore, as AG is to DK, so is AB to DE: but AG is equal to C, and DK to F: therefore, as C is to F, so is AB to DE. Therefore magnitudes, &c. Q. E. D. * 7. 5. * 12. 5.

PROP. XVI. THEOR.

If four magnitudes of the same kind *be proportionals, they shall also be proportionals when. taken alternately.*

Let the four magnitudes of the same kind A, B, C, D be proportionals, viz. as A to B, so C to D: they shall also be proportionals when taken alternately; that is, A shall be to C, as B to D.

Take of A and B any equimultiples whatever E and F; and of C and D any equimultiples whatever G and H: and because E is the same multiple of A, that F is of B, and that magnitudes have the same ratio to
* 15. 5. one another which their equimultiples have;* therefore A is to B, as E is to F: but as A is to B, so is C
* Hyp. to D:* therefore as C is to
* 11. 5. D, so* is E to F. Again, because G, H are equimultiples of C, D, therefore as
* 15. 5. C is to D, so is G to H;*



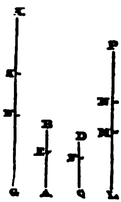
but it was proved that as C is to D, so is E to F;
* 11. 5. therefore, as E is to F, so is G to H.* But when four magnitudes are proportionals, if the first be greater than the third, the second is greater than the fourth;
* 14. 5. and if equal, equal; and if less, less:* therefore, if E be greater than G, F is greater than H: and if equal, equal; and if less, less: and E, F are any equimul-
* Const. tiples* whatever of A, B; and G, H any whatever of
* 5 Def. 5. C, D: therefore A is to C, as B to D.* If, then, four magnitudes, &c. Q. E. D.

PROP. XVII. THEOR.

If magnitudes, taken jointly, be proportionals, they shall also be proportionals when taken separately: that is, if two magnitudes together, have to one of them, the same ratio which two others have to one of these, the remaining one of the first two shall have to the other, the same ratio which the remaining one of the last two has to the other of these.

Let AB, BE, CD, DF be the magnitudes, taken jointly, which are proportionals; that is, as AB to BE, so is CD to DF; they shall also be proportionals taken separately, viz. AE shall be to EB, as CF to FD.

Take of AE, EB, CF, FD any equimultiples whatever GH, HK, LM, MN; and again, of EB, FD, take any equimultiples whatever KX, NP: then because GH is the same multiple of AE, that HK is of EB, therefore GH is the same multiple* of AE, that GK * 1. 5.
is of AB: but GH is the same multiple of AE, that LM is of CF; therefore GK is the same multiple of AB, that LM is of CF. Again, because LM is the same multiple of CF, that MN is of FD: therefore LM is the same multiple* of CF, that LN is of CD; but * 1. 5.
LM was shown to be the same multiple of CF, that GK is of AB; therefore GK is the same multiple of AB, that LN is of CD; that is, GK, LN, are equimultiples of AB, CD. Next, because HK is the same multiple of EB, that MN is of FD; and that KX is also the same multiple of EB, that NP is of FD; therefore HX is the same multiple* of EB, that MP * 2. 5.
is of FD. And because AB is to BE,



*as CD is to DF,** and that of AB and CD, GK and LN * Hyp.
are equimultiples, and of EB and FD HX and MP

are equimultiples; therefore if GK be greater than HX, LN is greater than MP; and if equal, equal; and if less, less:* but if GH be greater than KX, then by adding the common part HK to both, GK is greater than HX;* therefore LN is greater than MP; and by taking away MN from both, LM is greater* than NP: therefore, if GH be greater than KX, LM is greater than NP. In like manner it may be demonstrated, that if GH be equal to KX, LM is equal to NP; and if less, less: but GH, LM are any equimultiples whatever of AE, CF,* and KX, NP are any whatever of EB, FD: therefore,* as AE is to EB, so is CF to FD. If then, magnitudes, &c. Q. E. D.

* 5 Def. 5.
* 4 Ax. 1.
* 5 Ax. 1.
* Const.
* 5 Def. 5.

PROP. XVIII. THEOR.

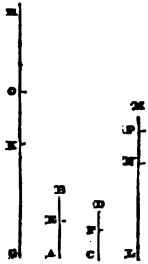
If magnitudes, taken separately, be proportionals, they shall also be proportionals when taken jointly: that is, if the first be to the second, as the third to the fourth, the first and second together, shall be to the second, as the third and fourth together, to the fourth.

Let AE, EB, CF, FD be proportionals; that is, as AE to EB, so is CF to FD; they shall also be proportionals when taken jointly; that is, AB shall be to BE, as CD to DF.

Take of AB, BE, CD, DF any equimultiples whatever GH, HK, LM, MN; and again, of BE, DF, take any equimultiples whatever KO, NP: and because KO, NP are equimultiples of BE, DF; and that KH NM are likewise equimultiples of BE, DF, therefore if KO, the multiple of BE, be greater than KH, which is a multiple of the same BE, then NP, the multiple of DF, is greater than MN, the multiple of the same DF; and if KO be equal to KH, NP is equal to NM; and if less, less.

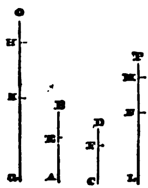
First, let KO be not greater than KH, therefore

NP is not greater than NM: and because GH, HK are equimultiples of AB, BE, and that AB is greater than BE, therefore GH is greater* than HK; but KO is not greater than KH, therefore GH is greater than KO. In like manner it may be proved, that LM is greater than NP: therefore, if KO be not greater than KH, then GH, the multiple of AB, is always greater than KO, the multiple of BE; and likewise LM, the multiple of CD, is greater than NP, the multiple of DF.



* 3 Ax. 5.

Next, let KO be greater than KH: therefore, as has been shown, NP is greater than NM: and because the whole GH is the same multiple of the whole AB, that HK is of BE, therefore the remainder GK is the same multiple of the remainder AE that GH is of AB:* which is the same that LM is of CD. In like manner, because LM is the same multiple of CD, that MN is of DF, the remainder LN is the same multiple of the remainder CF, that the whole LM is of the whole CD:* but it was shown that LM is the same multiple of CD, that GK is of AE; therefore GK is the same multiple of AE, that LN is of CF; that is, GK, LN are equimultiples of AE, CF: and because KO, NP are equimultiples of BE, DF, and if from KO, NP there be taken KH, NM, which are likewise equimultiples of BE, DF, therefore the remainders HO, MP are either equal to BE, DF, or equimultiples of them.* First, let HO, MP, be equal to BE, DF; then because AE *is to EB,* as CF to FD,* and that GK, LN are equi*multiples of AE, CF;* therefore GK is to EB, as LN



* 5. 5.

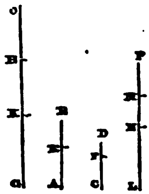
* 5. 5.

* 6. 5.

* Hyp.

* Cor. 4. 5. to FD:* but HO is equal to EB, and MP to FD; therefore GK is to HO, as LN to MP: therefore if
* A. 5. GK be greater than HO, LN is greater* than MP; and if equal, equal; and if less, less.

But let HO, MP be equimultiples of EB, FD; then
* Hyp. because* AE is to EB, as CF to FD, and that of AE, CF are taken equimultiples GK, LN; and of EB, FD, the equimultiples HO, MP; therefore if GK be greater than HO, LN is greater than MP; and if equal, equal; and if less,
* 5 Def. 5. less;* which was likewise shown in the preceding case. But if GH be greater than KO, taking KH from
* 5 Ax. 1. both, then GK is greater* than HO; therefore LN is greater than MP; and consequently, adding NM to
* 4 Ax. 1. both, LM is greater* than NP: therefore, if GH be greater than KO, LM is greater than NP. In like manner it may be shown, that if GH be equal to KO, LM is equal to NP; and if less, less. And in the case in which KO is not greater than KH, it has been shown that GH is always greater than KO, and LM greater than NP: but GH, LM are any equi-
* Const. multiples* of AB, CD, and KO, NP are any whatever
* 5 Def. 5. of BE, DF; therefore,* as AB is to BE, so is CD to DF. If, then, magnitudes, &c. Q. E. D.



PROP. XIX. THEOR.

If a whole magnitude be to a whole, as a magnitude taken from the first is to a magnitude taken from the other; the remainder shall be to the remainder, as the whole to the whole.

Let the whole AB be to the whole CD, as AE a magnitude taken from AB, is to CF a magnitude taken

from CD; the remainder EB shall be to the remainder FD, as the whole AB to the whole CD.

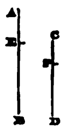
Because AB is to CD, as AE to CF; therefore, alternately,* BA is to AE, as DC to CF; and because * 16. 5.
if magnitudes taken jointly be proportionals, they are also proportionals* when taken separately; * 17. 5.
therefore, as BE is to EA, so is DF to FC; and alternately, as BE is to DF, so is EA to FC: but, by hypothesis, as AE is to CF, so is AB to CD; therefore the remainder BE is to the remainder DF, as the whole AB to the whole CD.* Wherefore, if the whole, &c. Q. E. D. * 11. 5.

Cor. If the whole be to the whole, as a magnitude taken from the first, is to a magnitude taken from the other; the remainder shall be to the remainder, as the magnitude taken from the first to that taken from the other. The demonstration is contained in the preceding.

PROP. E. THEOR.

If four magnitudes be proportionals, they are also proportionals by conversion; that is, the first is to its excess above the second, as the third to its excess above the fourth.

Let AB be to BE, as CD to DF; then BA shall be to AE, as DC to CF.



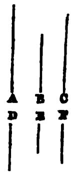
Because AB is to BE, as CD to DF, therefore by division,* AE is to EB, as CF to FD; * 17. 5.
and by inversion,* BE is to EA, as DF to FC; * B. 5.
therefore, by composition,* BA is to AE, as DC is to CF. If, therefore, four, &c. Q. E. D. * 18. 5.

PROP. XX. THEOR.

If there be three magnitudes, and other three, which, taken two and two, have the same ratio; then if the first

be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

Let A, B, C be three magnitudes, and D, E, F other three, which, taken two and two, have the same ratio, viz. as A is to B, so is D to E; and as B to C, so is E to F: then if A be greater than C, D shall be greater than F; and if equal, equal; and if less, less.



Because A is greater than C, and B is any other magnitude, and that the greater has to the same magnitude a greater ratio
* 8. 5. than the less has to it;* therefore A has to
* Hyp. B a greater ratio than C has to B: but* as D
* 13. 5. is to E, so is A to B; therefore* D has to E a greater ratio than C has to B; and because B is to C, as E to
* B. 5. F, by inversion,* C is to B, as F is to E; and D was shown to have to E a greater ratio than C to B; there-
* Cor.13. 5. fore D has to E a greater ratio than F to E:* but the magnitude which has a greater ratio than another
* 10. 5. to the same magnitude, is the greater* of the two: therefore D is greater than F.

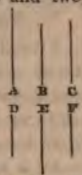
Secondly, let A be equal to C; D shall be equal to F. Because A and C are equal to one another, A is to B, as C is to
* 7. 5. B:* but A is to B, as D to E; and C is to B, as F to E; wherefore D
* 11. 5. is to E, as F to E;* and therefore
* 9. 5. D is equal to F.*

Next, let A be less than C; D shall be less than F. For C is greater than A, and, as was shown in the first case, C is to B, as F is to E, and in like manner B is to A, as E is to D: therefore by the first case F is greater than D; and therefore D is less than F. Therefore, if there be three, &c. *Q. E. D.*

PROP. XXI. THEOR.

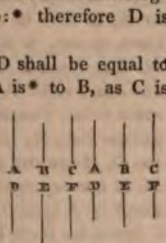
If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order; then if the first magnitude be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

Let A, B, C be three magnitudes, and D, E, F other three, which have the same ratio, taken two and two, but in a cross order, viz, as A is to B, so is E to F, and as B is to C, so is D to E: then if A be greater than C, D shall be greater than F; and if equal, equal; and if less, less.



Because A is greater than C, and B is any other magnitude, A has to B a greater ratio* than C has to B: but as E to F, so is A to B: * 8. 5.
therefore* E has to F a greater ratio than C has to B: * 13. 5.
and because* B is to C, as D to E, by inversion, C is to * Hyp.
B, as E to D: and it was shown that E has to F a greater ratio than C to B; therefore E has to F a greater ratio than E to D;* but the magnitude which * Cor. 13. 5.
has a greater ratio than another has to the same magnitude, is the greater of the two:* therefore D is * 10. 5.
greater than F.

Secondly, let A be equal to C; D shall be equal to F. Because A and C are equal, A is* to B, as C is * 7. 5.
to B: but* A is to B, as E to F; * Hyp.
and C is to B, as E to D; therefore E is to F as E to D;* and * 11. 5.
therefore D is equal* to F. * 9. 5.



Next, let A be less than C; D shall be less than F. For C is greater than A, and, as was shown, C is to B, as E is to D, and in like manner B is to A, as F is to E;

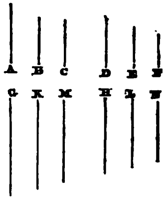
therefore by the first case F is greater than D, that is, D is less than F. Therefore, if there be three, &c. Q. E. D.

PROP. XXII. THEOR.

If there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio; the first shall have to the last of the first magnitudes, the same ratio which the first has to the last of the others. N. B. This is usually cited by the words "ex æquali," or "ex æquo."

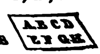
First, let there be three magnitudes A, B, C, and as many others D, E, F, which taken two and two in order, have the same ratio, that is, such, that A is to B as D to E: and as B is to C, so is E to F; then A shall be to C, as D to F.

Of A and D take any equimultiples whatever G and H; and of B and E any equimultiples whatever K and L; and of C and F any whatever M and N: then because A is to B, as D to E, and that G, H are equimultiples of A, D, and K, L equimultiples of B, E; therefore as G is
* 4. 5. to K, so is* H to L. For the same reason, K is to M, as L to



N: and because there are three magnitudes G, K, M, and other three H, L, N, which, two and two, have the same ratio; therefore if G be greater than M, H is
* 20 5. greater than N; and if equal, equal; and if less, less;*
* Const. but G, H are any equimultiples* whatever of A, D, and M, N are any equimultiples whatever of C, F: there-
* 5 Def. 5. fore,* as A is to C, so is D to F.

Next, let there be four magnitudes, A, B, C, D, and other four, E, F, G, H, which two and two have the same ratio, viz. as A is to B, so is



E to F; and as B to C, so F to G; and as C to D, so G to H: then A shall be to D, as E to H.

Because A, B, C are three magnitudes, and E, F, G other three, which, taken two and two, have the same ratio; therefore, by the foregoing case, A is to C, as E to G: but C is to D, as G is to H; and since there are three magnitudes A, C, D, and three others E, G, H, which taken two and two have the same ratio; therefore, by the first case, A is to D, as E to H; and so on, whatever be the number of magnitudes. Therefore, if there be any number, &c. Q. E. D.

PROP. XXIII. THEOR.

If there be any number of magnitudes, and as many others, which taken two and two in a cross order, have the same ratio: the first shall have to the last of the first magnitudes, the same ratio which the first has to the last of the others. N. B. This is usually cited by the words, "ex æquali in proportione perturbata;" or "ex æquo perturbato."

First, let there be three magnitudes A, B, C, and other three D, E, F, which taken two and two in a cross order have the same ratio, that is, such, that A is to B, as E to F; and as B is to C, so is D to E: then A shall be to C, as D to F.

Take of A, B, D any equimultiples whatever G, H, K; and of C, E, F any equimultiples whatever L, M, N; and because G, H are equimultiples of A, B, and that magnitudes have the same ratio which their equimultiples have;* * 15. 5.
therefore as A is to B, so is G to H: and for the same reason, as E is to F, so is M to N: but as A is to B,* so is E to F; * Hyp.
therefore as G is to H, so is M to N.* And because as B is to * 11. 5.

* Hyp. C,* so is D to E, and that H, K are equimultiples of B, D, and L, M of C, E; therefore as H is to L, so
* 4. 5. is* K to M: and it has been shown that G is to H, as M to N: then, because there are three magnitudes G, H, L, and other three K, M, N which have the same ratio taken two and two in a cross order; therefore if G be greater than L, K is greater than N; and if
* 21. 5. equal, equal; and if less, less;* but G, K are any equi-
* Const. multiples whatever* of A, D; and L, N any whatever
* 5 Def. 5. of C, F; therefore* as A is to C, so is D to F.

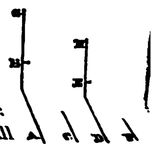
Next, let there be four magnitudes A, B, C, D, and other four E, F, G, H, which taken two and two in a cross order, have the same ratio, viz. A to B, as G to H; B to C, as F to G; and C to D, as E to F: then A shall be to D, as E to H.

A	B	C	D
E	F	G	H

Because A, B, C, are three magnitudes, and F, G, H other three, which, taken two and two in a cross order, have the same ratio; therefore, by the first case, A is to C, as F to H: but C is to D, as E is to F; therefore again, by the first case, A is to D, as E to H: and so on whatever be the number of magnitudes. Therefore, if there be any number, &c. Q. E. D.

PROP. XXIV. THEOR.

If the first has to the second the same ratio which the third has to the fourth; and the fifth to the second the same ratio which the sixth has to the fourth; the first and fifth together shall have to the second, the same ratio which the third and sixth together have to the fourth.



Let AB the first have to C the second, the same ratio which DE the third has to F the fourth; and let BG the fifth have to C the second, the same ratio which EH the sixth has to F the fourth: AG, the first and fifth together, shall

have to C the second, the same ratio which DH, the third and sixth together, has to F the fourth.

Because BG is to C, as EH to F; by inversion,* C is to BG, as F to EH: and because, as AB is to C, so is DE to F;* and as C to BG, so F to EH; therefore, ex æquali,* AB is to BG, as DE to EH: and because these magnitudes are proportionals, they are likewise proportionals when taken* jointly; therefore as AG is to GB, so is DH to HE; but* as GB to C, so is HE to F: therefore, ex æquali,* as AG is to C, so is DH to F. Wherefore, if the first, &c. Q. E. D.

* B. 5.
* Hyp.
* 22. 5.
* 18. 5.
* Hyp.
* 22. 5.

COR. 1. If the same hypothesis be made as in the proposition, the excess of the first and fifth shall be to the second, as the excess of the third and sixth to the fourth. The demonstration of this is the same with that of the proposition, if division be used instead of composition.

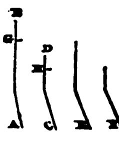
COR. 2. The proposition holds true of two ranks of magnitudes, whatever be their number, of which each of the first rank has to the second magnitude the same ratio that the corresponding one of the second rank has to a fourth magnitude; as is manifest.

PROP. XXV. THEOR.

If four magnitudes of the same kind are proportionals, the greatest and least of them together, are greater than the other two together.

Let the four magnitudes AB, CD, E, F be proportionals, viz. AB to CD, as E to F; and let AB be the greatest of them, and consequently F the least:* then AB together with F, shall be greater than CD together with E.

* A. and 14. 5.



Take AG equal to E, and CH equal to F: then because as AB is to CD, so is E to F, *and that AG is* equal to E, *and CH equal to F;* therefore AB is to

CD, as AG to CH: and because the whole AB, is to the whole CD, as AG is to CH: therefore the remainder GB is to the remainder HD, as the whole
* 19. 5. AB is to the whole* CD: but AB is greater than CD,
* A. 5. therefore* GB is greater than HD: and because AG is equal to E, and CH equal to F; therefore AG and
* 2 Ax. 1. F together, are equal* to CH and E together: now, if to the unequal magnitudes GB, HD, of which GB is the greater, there be added equal magnitudes, viz. to GB the two AG and F, and to HD the two CH and E;
* 4 Ax. 1. therefore AB and F together are greater* than CD and E. Therefore, if four magnitudes, &c. Q. E. D.

PROP. F. THEOR.

Ratios which are compounded of the same ratios, are equal to one another.

Let A be to B, as D to E; and B to C, as E to F: the ratio which is compounded of the ratios of A to B, and B to C, which, by the definition of compound ratio, is the ratio of A to C, shall be the same with the ratio of D to F, which, by the same definition, is compounded of the ratios of D to E, and E to F.

A	B	C
D	E	F

Because there are three magnitudes A, B, C, and three others D, E, F, which, taken two and two in order,
* 22. 5. have the same ratio; ex æquali, A is to C, as D to F.*

Next, let A be to B, as E to F, and B to C, as D to
* 23. 5. E; therefore, *ex æquali in proportione perturbata,** A is to C, as D to F; that is, the ratio of A to C, which is compounded of the ratios of A to B, and B to C, is the same with the ratio of D to F, which is compounded of the ratios of D to E, and E to F. In like manner the proposition may be demonstrated whatever be the number of ratios in either case.

A	B	C
D	E	F

PROP. G. THEOR.

If several ratios be the same with several ratios, each to

each; the ratio which is compounded of ratios which are the same with the first ratios, each to each, is the same with the ratio compounded of ratios which are the same with the other ratios, each to each.

Let A be to B as E to F; and C to D, as G to H: and let A be to B, as K to L; and C to D, as L to M: then the ratio of K to M, by the definition of compound ratio, is compounded of the ratios of K to L, and L to M, which are the same with the ratios of A to B, and C to D: again, as E to F, so let N be to O; and as G to H, so let O be to P; then the ratio of N to P is compounded of the ratios of N to O, and O to P, which are the same with the ratios of E to F, and G to H: and it is to be shown that the ratio of K to M, shall be the same with the ratio of N to P, or that K shall be to M, as N to P.

A	B	C	D	K	L	M
E	F	G	H	N	O	P

Because K is to L, as (A to B, that is, as E to F, that is, as) N to O; and as L to M, so is (C to D, and so is G to H, and so is) O to P: therefore ex æquali* K is to M, as N to P. Therefore, if several ratios, &c. Q. E. D. * 22. 5.

PROP. H. THEOR.

If a ratio compounded of several ratios be the same with a ratio compounded of any other ratios, and if one of the first ratios, or a ratio compounded of any of the first, be the same with one of the last ratios, or with the ratio compounded of any of the last; then the ratio compounded of the remaining ratios of the first, or the remaining ratio of the first, if but one remain, is the same with the ratio compounded of those remaining of the last, or with the remaining ratio of the last.

Let the first ratios be those of A to B, B to C, C to

D, D to E, and E to F; and let the other ratios be those of G to H, H to K, K to L, and L to M: also, let the ratio of A to F, which is compounded of * the first ratios, be the same with the ratio of G to M, which is compounded of the other ratios: and besides, let the ratio of A to D, which is compounded of the ratios of A to B, B to C, C to D, be the same with the ratio of G to K, which is compounded of the ratios of G to H, and H to K: then the ratio compounded of the remaining first ratios, to wit, of the ratios of D to E and E to F, which compounded ratio is the ratio of D to F, shall be the same with the ratio of K to M, which is compounded of the remaining ratios of K to L, and L to M of the other ratios.

* Definition of compounded ratio.

A B C D E F
G H K L M

Because, by the hypothesis, A is to D, as G to K,
* B. 5. by inversion,* D is to A, as K to G; but as A is to F,
* 22. 5. so is G to M; therefore,* ex æquali, D is to F, as K to M. Therefore, if a ratio which is, &c. Q. E. D.

PROP. K. THEOR.

If there be any number of ratios, and any number of other ratios such, that the ratio compounded of ratios which are the same with the first ratios, each to each, is the same with the ratio compounded of ratios which are the same, each to each, with the last ratios; and if one of the first ratios, or the ratio which is compounded of ratios which are the same with several of the first ratios, each to each, be the same with one of the last ratios, or with the ratio compounded of ratios which are the same, each to each, with several of the last ratios: then the ratio compounded of ratios which are the same with the remaining ratios of the first, each to each, or the remaining ratio of the first, if but one remain; is the same with the ratio com-

pounded of ratios which are the same with those remaining of the last, each to each, or with the remaining ratio of the last.

Let the ratios of A to B, C to D, E to F be the first ratios; and the ratios of G to H, K to L, M to N, O to P, Q to R, be the other ratios: and let A be to B, as S to T; and C to D, as T to V, and E to F, as V to X; then, by the definition of compound ratio, the ratio

h, k, l.
A, B; C, D; E, F. S, T, V, X.
G, H; K, L; M, N; O, P; Q, R. Y, Z, a, b, c, d.
e, f, g. m, n, o, p.

of S to X is compounded of the ratios of S to T, T to V, and V to X, which are the same with the ratios of A to B, C to D, E to F, each to each; also, as G to H, so let Y be to Z; and K to L, as Z to a; M to N, as a to b, O to P, as b to c; and Q to R, as c to d: then, by the same definition, the ratio of Y to d is compounded of the ratios of Y to Z, Z to a, a to b, b to c, and c to d, which are the same, each to each, with the ratios of G to H, K to L, M to N, O to P, and Q to R: therefore, by the hypothesis, S is to X, as Y to d. Also, let the ratio of A to B, that is, the ratio of S to T, which is one of the first ratios, be the same with the ratio of e to g, which is compounded of the ratios of e to f, and f to g, which, by the hypothesis, are the same with the ratios of G to H, and K to L, two of the other ratios; and let the ratio of h to l be that which is compounded of the ratios of h to k, and k to l, which are the same with the remaining first ratios, viz. of C to D, and E to F; also, let the ratio of m to p, be that which is compounded of the ratios of m to n, n to o, and o to p, which are the same, each to each, with the

remaining other ratios, viz. of M to N, O to P, and Q to R: then the ratio of h to l shall be the same with the ratio of m to p, or h shall be to l, as m to p.

	h, k, l.	
	A, B; C, D; E, F.	S, T, V, X.
G, H; K, L; M, N; O, P; Q, R.		Y, Z, a, b, c, d.
e, f, g.	m, n, o, p.	

Because e is to f, as (G to H, that is, as) Y to Z;
and f is to g, as (K to L, that is, as) Z to a; therefore,
* 22. 5. ex æquali,* e is to g, as Y to a: but by the hypothesis,
* 11 5. A is to B, that is, S to T, as e to g; therefore* S is to
* B. 5. T, as Y to a; and, by inversion,* T is to S, as a to Y;
and S is to X, as Y to d; therefore, ex æquali; T is to X as a to d. Also, because h is to k as (C to D, that is, as) T to V; and k is to l, as (E to F, that is, as) V to X; therefore, ex æquali, h is to l, as T to X. In like manner, it may be demonstrated, that m is to p, as a to d: but it has been shown, that T is to X, as a
* 11. 5. to d; therefore* h is to l, as m to p. Q. E. D.

The propositions G and K are usually, for the sake of brevity, expressed in the same terms with propositions F and H: and therefore it was proper to show the true meaning of them when they are so expressed; especially since they are very frequently made use of by geometers.

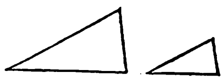
THE

ELEMENTS OF EUCLID.

BOOK VI.

DEFINITIONS.

I.



Similar rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportionals.

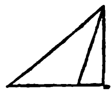
II.

"*Reciprocal figures*, viz. triangles and parallelograms, "are such as have their sides about two of their "angles proportionals in such a manner, that a side "of the first figure is to a side of the other, as the "remaining side of this other is to the remaining "side of the first."

III.

A straight line *is said to* be cut in *extreme and mean ratio, when the* whole is to the greater segment, as *the greater segment* is to the less.

IV.



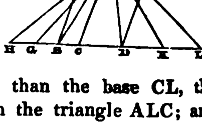
The *altitude* of any figure is the straight line drawn from its vertex perpendicular to the base.

PROP. I. THEOR.

Triangles and parallelograms of the same altitude are one to another as their bases.

Let the triangles ABC, ACD, and the parallelograms EC, CF have the same altitude, viz. the perpendicular drawn from the point A to BD, or BD produced: then, as the base BC is to the base CD, so shall the triangle ABC be to the triangle ACD, and the parallelogram EC to the parallelogram CF.

* 3. 1. Produce BD both ways, and take* any number of straight lines BG, GH, each equal to the base BC; and DK, KL, any number of them, each equal to the base CD; and join AG, AH, AK, AL: then, because CB, BG, GH are all equal, the triangles AHG, AGB,
* 38. 1. ABC are all equal:* therefore, whatever multiple the base CH is of the base BC, the same multiple is the triangle AHC of the triangle ABC: for the same reason, whatever multiple the base CL is of the base CD, the same multiple is the triangle ALC of the triangle ADC: and if the base HC be equal to the base CL,
* 38 1. the triangle AHC is equal* to the triangle ALC; and if the base HC be greater than the base CL, the triangle AHC is greater than the triangle ALC; and if less, less: therefore, since there are four magnitudes, *viz.* the two bases BC, CD, and the two triangles ABC, ACD; and of the base BC and the triangle ABC, the first and third, any equimultiples whatever have been



taken, viz. the base HC and triangle AHC; and of the base CD and triangle ACD, the second and fourth, have been taken any equimultiples whatever, viz. the base CL and triangle ALC; and that it has been shown, that, if the base HC be greater than the base CL, the triangle AHC is greater than the triangle ALC; and if equal, equal; and if less, less: therefore,* as the base BC is to the base CD, so is the triangle ABC to the triangle ACD. * 5 Def. 5.

And because the parallelogram CE is double of the triangle ABC,* and the parallelogram CF double of the triangle ACD, and that magnitudes have the same ratio which their equimultiples have;* as the triangle ABC is to the triangle ACD, so is the parallelogram EC to the parallelogram CF: and because it has been shown, that, as the base BC is to the base CD, so is the triangle ABC to the triangle ACD: and as the triangle ABC is to the triangle ACD, so is the parallelogram EC to the parallelogram CF; therefore, as the base BC is to the base CD, so is* the parallelogram EC to the parallelogram CF. Wherefore triangles, &c. Q. E. D. * 41. 1. * 15. 5. * 11. 5.

Cor. From this it is plain, that triangles and parallelograms that have equal altitudes, are one to another as their bases.

Let the figures be placed so as to have their bases in the same straight line; and having drawn perpendiculars from the vertices of the triangles to the bases, the straight line which joins the vertices is parallel to that in which their bases are,* because the perpendiculars are both equal and parallel* to one another. Then, if the same construction be made as in the proposition, the demonstration will be the same. * 33. 1 * 28. 1

PROP. II. THEOR.

If a straight line be drawn parallel to one of the sides

of a triangle, it shall cut the other sides, or these produced, proportionally.

And if the sides, or the sides produced, be cut proportionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.

Let DE be drawn parallel to BC, one of the sides of the triangle ABC: then BD shall be to DA, as CE to EA.

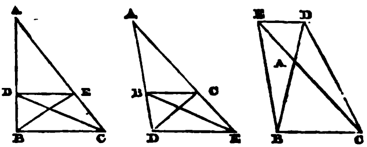
* 37. 1. Join BE, CD; then the triangle BDE is equal* to the triangle CDE, because they are on the same base DE, and between the same parallels DE, BC: but ADE is another triangle, and equal magnitudes have

* 7. 5. the same ratio to the same magnitude;* therefore, as the triangle BDE to the triangle ADE, so is the triangle CDE to the triangle ADE; but as the triangle BDE

* 1. 6. to the triangle ADE, so is* BD to DA, because having the same altitude, viz. the perpendicular drawn from the point E to AB, they are to one another as their bases; and for the same reason, as the triangle CDE to the triangle ADE, so is CE to EA: therefore, as

* 11. 5. BD to DA, so is CE to EA.*

Next, let the sides AB, AC of the triangle ABC, or



these produced, be cut proportionally in the points D, E, that is, so that BD be to DA, as CE to EA, and join DE; DE shall be parallel to BC.

The same construction being made, because as BD to DA, so is CE to EA; and as BD to DA, so is the

* 1. 6. triangle BDE to the triangle ADE;* and as CE to

EA, so is the triangle CDE to the triangle ADE; therefore* the triangle BDE is to the triangle ADE, as the triangle CDE to the triangle ADE; that is, the triangles BDE, CDE have the same ratio to the triangle ADE; and therefore* the triangle BDE is equal to the triangle CDE: and they are on the same base DE; but equal triangles on the same base, and on the same side of it, are between the same parallels;* therefore DE is parallel to BC. Wherefore, if a straight line, &c. Q. E. D.

* 11. 5.
* 9. 5.
* 39. 1

PROP. III. THEOR.

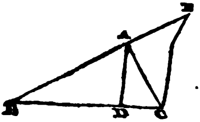
If the angle of a triangle be divided into two equal angles, by a straight line which also cuts the base; the segments of the base shall have the same ratio which the other sides of the triangle have to one another.

And if the segments of the base have the same ratio which the other sides of the triangle have to one another, the straight line drawn from the vertex to the point of section, divides the vertical angle into two equal angles.

Let ABC be any triangle, and let the angle BAC be divided into two equal angles by the straight line AD: BD shall be to DC, as BA to AC.

Through the point C draw CE parallel* to DA, and let BA produced meet CE in E. Because the straight line AC meets the parallels AD, EC, the angle ACE is equal* to the alternate angle CAD: but the angle CAD, by the hypothesis, is equal to the angle BAD; wherefore the angle BAD is equal* to the angle ACE. Again, because the straight line BAE meets the parallels AD, EC, the outward angle BAD is equal* to the inward and opposite angle AEC: but

* 31. 1.
* 29. 1.
* 1 Ax.
* 29. 1.



the angle ACE has been proved equal to the angle BAD; therefore ACE is also equal to the angle AEC,
* 6. 1. and consequently the side AE is equal* to the side AC: and because AD is drawn parallel to EC, one of the sides of the triangle BCE, therefore BD is to DC,
* 2. 6. as BA to AE;* but AE has been proved equal to AC;
* 7. 5. therefore, as BD to DC, so is BA to AC.*

Next, let BD be to DC, as BA to AC, and join AD; the angle BAC shall be divided into two equal angles by the straight line AD.

The same construction being made; because, as BD to DC, so is BA to AC; and as BD to DC, so is BA
* 2. 6. to AE,* because AD is parallel to EC; therefore BA
* 11. 5. is to AC, as BA to AE:* consequently AC is equal to
* 9. 5. AE,* and therefore the angle AEC is equal to the
* 5. 1. angle ACE:* but the angle AEC is equal to the outward and opposite angle BAD; and the angle ACE is
* 29. 1. equal* to the alternate angle CAD: therefore also the
* 1 Ax. angle BAD is equal* to the angle CAD; that is, the angle BAC is cut into two equal angles by the straight line AD. Therefore, if the angle, &c. Q. E. D.

PROP. A. THEOR.

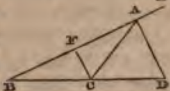
If the outward angle of a triangle made by producing one of its sides, be divided into two equal angles, by a straight line which also cuts the base produced; the segments between the dividing line and the extremities of the base, have the same ratio which the other sides of the triangle have to one another.

And if the segments of the base produced, have the same ratio which the other sides of the triangle have, the straight line drawn from the vertex to the point of section divides the outward angle of the triangle into two equal angles.

Let ABC be any triangle, and let the outward angle

CAE be divided into two equal angles by the straight line AD which meets the base produced in D: BD shall be to DC, as BA to AC.

Through C draw* CF parallel to DA; and because * 31. 1.
the straight line AC meets the parallels AD, FC, the angle ACF is equal* to the alternate angle CAD: but * 29. 1.
CAD is equal* to the angle DAE; therefore DAE is * Hyp.
equal* to the angle ACF. Again, because the straight * 1 Ax.
line FAE meets the parallels AD, FC, the outward angle DAE is equal* to the in- * 29. 1.
ward and opposite angle CFA: but the angle ACF has been proved equal to the angle DAE;



therefore the angle ACF is equal* * 1 Ax.
to the angle CFA, and consequently the side AF is equal* to the side AC: and because AD is parallel to * 6. 1.
FC, a side of the triangle BCF, therefore BD is to DC,* as BA to AF; but AF has been proved equal to * 2. 6.
AC; therefore as BD is to DC,* so is BA to AC. * 7. 5.

Next, let BD be to DC, as BA to AC, and join AD; the angle CAD shall be equal to the angle DAE.

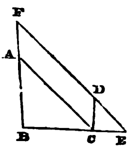
The same construction being made, because BD is to DC, as BA to AC; and also that BD is to DC,* as * 2. 6.
BA to AF; therefore* BA is to AC, as BA to AF; * 11. 5.
wherefore AC is equal* to AF, and the angle AFC * 9. 5.
equal* to the angle ACF: but the angle AFC is equal * 5. 1.
to the outward angle EAD,* and the angle ACF to the * 29. 1.
alternate angle CAD; therefore EAD is equal* to the * 1 Ax.
angle CAD. Wherefore, if the outward, &c. Q. E. D.

PROP. IV. THEOR.

The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides, that is, are the antecedents or consequents of the ratios.

Let ABC, DCE be equiangular triangles, having the angle ABC equal to the angle DCE, and the angle
* 32. 1. ACB to the angle DEC, and consequently* the angle BAC equal to the angle CDE: the sides about the equal angles of the triangles ABC, DCE shall be proportionals; and those shall be the homologous sides which are opposite to the equal angles.

* 22. 1. Let the triangle DCE be placed,* so that its side CE may be contiguous to BC, and in the same straight
* Hyp. line with it: then because the angle BCA is equal* to the angle CED, add the angle ABC to each; therefore
* 2 Ax. the two angles ABC, BCA are equal* to the two angles ABC, CED; but the two angles ABC, BCA are to-
* 17. 1. gether less* than two right angles, therefore the angles ABC and DEC, are less than two right angles; where-
* 12 Ax. 1. fore BA, ED if produced will meet;* let them be produced and meet in the point F: then because the angle ABC



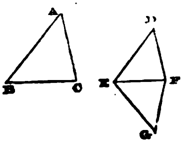
* Hyp. is equal* to the angle DCE, BF is
* 28. 1. parallel* to CD; and because the angle ACB is equal
* 28. 1. to the angle DEC, AC is parallel* to FE: therefore FACD is a parallelogram; and consequently AF is
* 34. 1. equal* to CD, and AC to FD: and because AC is parallel to FE, one of the sides of the triangle FBE,
* 2. 6. BA is to AF, as BC to CE:* but AF has been proved
* 7. 5. equal to CD; therefore,* as BA to CD, so is BC to
* 16. 5. CE; and alternately,* as AB to BC, so is DC to CE: again, because CD is parallel to BF, as BC to CE, so
* 2. 6. is FD to DE;* but FD has been proved equal to AC;
* 7. 5. therefore,* as BC to CE, so is AC to DE: and alter-
* 16. 5. nately,* as BC to CA, so CE to DE: but it has been proved that AB is to BC, as DC to CE, and that BC
* 22. 5. is to CA as CE to ED; therefore, ex æquali,* BA is to *AC as CD* to DE. Therefore the sides, &c. Q. E. D.

PROP. V. THEOR.

If the sides of two triangles, about each of their angles, be proportionals, the triangles shall be equiangular, and the equal angles shall be those which are opposite to the homologous sides.

Let the triangles ABC, DEF have their sides proportionals, so that AB is to BC, as DE to EF; and BC to CA, as EF to FD; and consequently, ex æquali, BA to AC, as ED to DF; the triangle ABC shall be equiangular to the triangle DEF, and the angles which are opposite to the homologous sides shall be equal, viz. the angle ABC equal to the angle DEF, and BCA to EFD, and also BAC to EDF.

At the points E, F, in the straight line EF, make* * 23. 1.
the angle FEG equal to the angle ABC, and the angle EFG equal to BCA; wherefore the remaining angle EGF is equal to the remaining angle BAC,* and * 32. 1.
therefore the triangle GEF is equiangular to the triangle ABC; and consequently they have their sides opposite to the equal angles proportionals:* wherefore, as * 4. 6.
AB to BC, so is GE to EF; but as AB to BC,* so is DE to * Hyp.
EF; therefore as DE to EF, so* GE to EF: that is, DE and * 11. 5.



GE have the same ratio to EF, and consequently are equal:* for the same reason, DF is equal to FG: and * 9. 5.
because, in the triangles DEF, GEF, DE has been proved equal to EG, and EF is common, the two sides DE, EF are equal to the two GE, EF, each to each, and the base DF is equal to the base GF; therefore the angle DEF is equal* to the angle GEF, and the * 8. 1.
other angles to the *other* angles which are subtended *by the equal sides.** Therefore the angle DFE is equal * 4. 1.

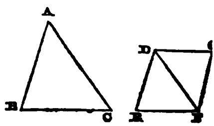
to the angle GFE, and EDF to EGF: and becaus
the angle DEF is equal to the angle GEF, and GE
* Const. to the angle ABC; * therefore the angle ABC is equa
to the angle DEF: for the same reason, the angl
ACB is equal to the angle DFE, and the angle at A t
the angle at D: therefore the triangle ABC is equi
angular to the triangle DEF. Wherefore, if the side
&c. Q. E. D.

PROP. VI. THEOR.

If two triangles have one angle of the one equal to on angle of the other, and the sides about the equ angles proportionals, the triangles shall be equiangula and shall have those angles equal which are opposi to the homologous sides.

Let the triangles ABC, DEF have the angle BA in the one equal to the angle EDF in the other, an the sides about those angles proportionals; that is, B. to AC, as ED to DF; the triangles ABC, DEF sha be equiangular, and shall have the angle ABC equ to the angle DEF, and ACB to DFE.

* 23. 1. At the points D, F, in the straight line DF, make the angle FDG equal to either of the angles BA(EDF; and the angle DFG equal to the angle ACB: wherefore the remaining angle at G is equal to the
* 32. 1. remaining angle at B,* and consequently the triangle



DGF is equiangular to the triangle ABC; and there
* 4. 6. fore as BA to AC, so is* GD to DF: but, by th
hypothesis, as BA to AC, so is ED to DF; therefor
* 11. 5. as ED to DF, so is* GD to DF; wherefore ED i
* 9. 5. equal* to DG; and DF is common to the two triangl
EDF, GDF: therefore the two sides ED DF w

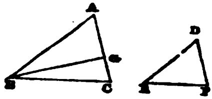
equal to the two sides GD, DF, each to each; and the angle EDF is equal* to the angle GDF; wherefore the * Const. base EF is equal* to the base FG, and the triangle * 4. 1. EDF to the triangle GDF, and the remaining angles to the remaining angles, each to each, which are subtended by the equal sides: therefore the angle DFG is equal to the angle DFE, and the angle at G to the angle at E: but the angle DFG is equal* to the angle * Const. ACB; therefore the angle ACB is equal* to the angle * 1 Ax. DFE: and the angle BAC is equal* to the angle EDF; * Hyp. wherefore the remaining angle at B is equal* to the * 32. 1. remaining angle at E. Therefore the triangle ABC is equiangular to the triangle DEF. Wherefore, if two triangles, &c. Q. E. D.

PROP. VII. THEOR.

If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals, then, if each of the remaining angles be either less, or not less, than a right angle; or if one of them be a right angle; the triangles shall be equiangular, and shall have those angles equal about which the sides are proportionals.

Let the two triangles ABC, DEF have one angle in the one equal to one angle in the other, viz. the angle BAC to the angle EDF, and the sides about two other angles ABC, DEF proportionals, so that AB is to BC, as DE to EF; and, in the first case, let each of the remaining angles at C, F be less than a right angle: the triangle ABC shall be equiangular to the triangle DEF, viz. the angle ABC shall be equal to the angle DEF, and the remaining angle at C to the remaining angle at F.

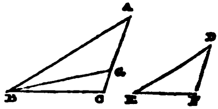
For, if the angles ABC, DEF be not equal, one of them must be greater than the other: let ABC be the



greater, and at the point B, in the straight line AB, make* the angle ABG equal to the angle DEF: and because the angle at A is equal* to the angle at D, and the angle ABG to the angle DEF; the remaining angle AGB is equal* to the remaining angle DFE: therefore the triangle ABG is equiangular to the triangle DEF; wherefore* as AB is to BG, so is DE to EF: but, by hypothesis, as DE to EF, so is AB to BC; therefore as AB to BC, so is AB to BG;* and because AB has the same ratio to each of the lines BC, BG: therefore BC is equal* to BG, and the angle BGC equal* to the angle BCG: but, by hypothesis, the angle BCG is less than a right angle; therefore the angle BGC is also less than a right angle, and therefore the adjacent angle AGB must be greater* than a right angle: but it was proved that the angle AGB is equal to the angle at F; therefore the angle at F is greater than a right angle: but this is absurd, because, by hypothesis, F is less than a right angle: therefore the angles ABC, DEF are not unequal, that is, they are equal: and the angle at A is equal* to the angle at D; wherefore the remaining angle at C is equal* to the remaining angle at F: therefore the triangle ABC is equiangular to the triangle DEF.

* 23. 1. * Hyp. * 32. 1. * 4. 6. * 11. 5. * 9. 5. * 5. 1. * 13. 1. * Hyp. * 32. 1.

Next, let each of the angles at C, F be not less than a right angle: the triangle ABC shall also in this case be equiangular to the triangle DEF.

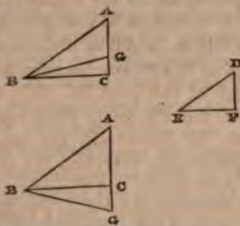


The same construction being made, it may be proved in like manner that BC is equal to BG, and the angle at C equal to the angle BGC; but the angle at C is not less* than a right angle; therefore the angle BGC is not less than a right

* Hyp.

angle: wherefore two angles of the triangle BGC are together not less than two right angles, which is impossible;* and therefore the triangle ABC may be proved to be equiangular to the triangle DEF, as in the first case. * 17. 1.

Lastly, let one of the angles at C, F, viz. the angle at C, be a right angle; in this case likewise the triangle ABC shall be equiangular to the triangle DEF.



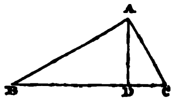
For, if they be not equiangular, at the point B in the straight line AB, make the angle ABG equal to the angle DEF; then it may be proved, as in the first case, that BG is equal to BC; and therefore* the angle BCG equal to the angle BGC; * 5. 1.
but the angle BCG is a right* angle, therefore the angle BGC is also a right angle: whence two of the angles of the triangle BGC are together not less than two right angles, which is impossible:* therefore the triangle ABC is equiangular to the triangle DEF. Wherefore, if two triangles, &c. Q. E. D. * Hyp. * 17. 1.

PROP. VIII. THEOR.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the base; the triangles on each side of it are similar to the whole triangle, and to one another.

Let ABC be a right-angled triangle, having the right angle BAC; and from the point A let AD be drawn perpendicular to the base BC: the triangles ABD, ADC shall be similar to the whole triangle ABC, and *to one another*.

* 11 Ax. Because the angle BAC is equal* to the angle ADB, each of them being a right angle, and that the angle at B is common to the two triangles ABC, ABD; the remaining angle ACB is equal* to the remaining angle BAD: therefore the triangle ABC is equiangular to the triangle ABD, and the sides about their equal angles are proportionals;* wherefore the triangles are similar:* in like manner it may be demonstrated, that the triangle ABC is equiangular and similar to the triangle ADC: and the triangles ABD, CDA, being both equiangular to ABC, are equiangular to each other; therefore the sides about the equal angles are proportionals,* and therefore the triangles ABD, ACD are equiangular and similar to each other. Therefore, in a right-angled, &c. Q. E. D.



* 32. 1.
* 4. 6
* 1 Def. 6.
* 4. 6.

Cor. From this it is manifest, that the perpendicular drawn from the right angle of a right-angled triangle to the base, is a mean proportional between the segments of the base; and also, that each of the sides is a mean proportional between the base, and the segment of it adjacent to that side. Because in the triangles BDA, ADC, BD is to DA, as DA to DC;* and in the triangles ABC, DBA, BC is to BA, as BA to BD;* and in the triangles ABC, ACD, BC is to CA, as CA to CD.*

* 4. 6.
* 4. 6.
* 4. 6.

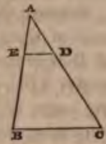
PROP. IX. PROB.

From a given straight line to cut off any part required.

Let AB be the given straight line; it is required to cut off any part from it.

From the point A draw a straight line AC making any angle with AB; and in AC take any point D, and take AC the same multiple of AD, that AB is of the

part which is to be cut off from it; join BC, and draw DE parallel to CB: then AE shall be the part required to be cut off.



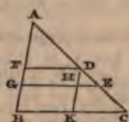
Because ED is parallel to BC, one of the sides of the triangle ABC, therefore as CD is to DA, so is* BE to EA; and, by composition,* CA is to AD, as BA to AE: but CA is a multiple of AD; therefore* BA is the same multiple of AE: whatever part therefore AD is of AC, AE is the same part of AB: wherefore, from the straight line AB the part required is cut off. Which was to be done. * 2. 6. * 18. 5. * D. 5.

PROP. X. PROB.

To divide a given straight line similarly to a given divided straight line, that is, into parts that shall have the same ratios to one another which the parts of the divided given straight line have.

Let AB be the straight line given to be divided, and AC the divided line; it is required to divide AB similarly to AC.

Let AC be divided in the points D, E; and let AB, AC be placed so as to contain any angle, and join BC, and through the points D, E, draw* DF, EG parallels to CB: then AB shall be divided in the points F, G similarly to AC. * 31. 1.



Through D draw DHK parallel to AB: therefore each of the figures FH, HB, is a parallelogram; wherefore DH is equal* to FG, and HK to GB: and because HE is parallel to KC, one of the sides of the triangle DKC, as CE to ED, so is* KH to HD: but KH is * 34. 1. * 2. 6.

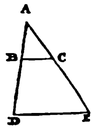
* 7. 5. equal to BG, and HD to GF; therefore,* as CE to ED, so is BG to GF: again, because FD is parallel to GE, one of the sides of the triangle AGE, as ED to DA,
* 2. 6. so is* GF to FA: therefore, as has been proved, CE is to ED, as BG to GF; and as ED to DA, so GF to FA. Therefore the given straight line AB is divided similarly to AC. Which was to be done.

PROP. XI. PROB.

To find a third proportional to two given straight lines.

Let AB, AC be the two given straight lines; it is required to find a third proportional to AB, AC.

Let AB, AC be placed so as to contain any angle; produce AB until BD, the part produced, is equal to AC; join BC, and through D,
* 31. 1. draw* DE parallel to BC, meeting AC produced in E: then CE shall be a third proportional to AB and AC.



Because BC is parallel to DE, a side
* 2. 6. of the triangle ADE, AB is* to BD, as AC to CE:
* 7. 5. but BD is equal to AC; therefore* as AB to AC, so is AC to CE: wherefore to the two given straight lines AB, AC a third proportional CE is found. Which was to be done.

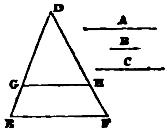
PROP. XII. PROB.

To find a fourth proportional to three given straight lines.

Let A, B, C be the three given straight lines; it is required to find a fourth proportional to A, B, C.

Take two straight lines DE, DF, containing any
* 3. 1. angle EDF; and upon these make* DG equal to A, GE equal to B, and DH equal to C; join GH, and

through E draw* EF parallel to it; then HF shall be the fourth proportional to A, B, C. * 31. 1.



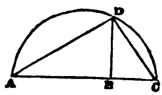
Because GH is parallel to EF, one of the sides of the triangle DEF, DG is to GE, as DH to HF;* but DG is equal to A, GE to B, and DH to C; * 2. 6. therefore,* as A is to B, so is C to HF: wherefore to * 7. 5. the three given straight lines, A, B, C a fourth proportional HF is found. Which was to be done.

PROP. XIII. PROB.

To find a mean proportional between two given straight lines.

Let AB, BC be the two given straight lines; it is required to find a mean proportional between them.

Place AB, BC in a straight line, and upon AC describe the semicircle ADC, and from the point B draw* BD at right angles to AC: then BD shall be a mean proportional between AB and BC. * 11. 1.



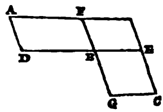
Join AD, DC; then the angle ADC, being in a semicircle is a right* angle; and because in the right-angled triangle ADC, DB is drawn from the right angle perpendicular to the base, therefore DB is a mean proportional between AB, BC the segments of the base:* therefore between the two given straight lines AB, BC, a mean proportional DB is found. Which was to be done. * 31. 3. * Cor. 8. 6.

PROP. XIV. THEOR.

Equal parallelograms which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: and parallelograms that have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Let AB, BC be equal parallelograms, which have the angles of B equal: the sides of the parallelograms AB, BC about the equal angles, shall be reciprocally proportional; that is, DB shall be to BE, as GB to BF.

Let the sides DB, BE be placed in the same straight
* 14. 1. line; wherefore also[a] FB, BG are in one straight line:* complete the parallelogram FE; and because the parallelogram AB is equal to BC, and that FE is another parallelo-



* 7. 5. gram, AB is to FE, as BC to FE:* but as AB to FE, so is the base
* 1. 6. DB to BE;* and, as BC to FE, so is the base GB to BF; therefore, as DB to BE, so is
* 11. 5. GB to BF;* wherefore the sides of the parallelograms AB, BC about their equal angles are reciprocally proportional.

Next, let the sides about the equal angles be reciprocally proportional, viz. as DB to BE, so GB to BF; the parallelogram AB shall be equal to the parallelogram BC.

* 1. 6. Because, as DB to BE, so is GB to BF; and* as DB

[a] Because the angles FBD, FBE are together equal to two
* 13. 1. right angles,* and that the angle DBF is equal to the angle EBG, by hypothesis; therefore the two angles FBE, EBG are
* 14. 1. together equal to two right angles, and consequently* FB, BG are in the same straight line.

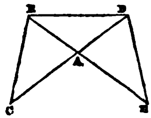
to BE, so is the parallelogram AB to the parallelogram FE; and as GB to BF, so is the parallelogram BC to the parallelogram FE; therefore* as AB to FE, so BC to FE; therefore the parallelogram AB is equal* to the parallelogram BC. Therefore equal parallelograms, &c. Q. E. D. * 11. 5. * 9. 5.

PROP. XV. THEOR.

Equal triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: and triangles which have one angle in the one equal to one angle in the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Let ABC, ADE be equal triangles, which have the angle BAC equal to the angle DAE; the sides about the equal angles of the triangles shall be reciprocally proportional; that is, CA shall be to AD, as EA to AB.

Let the triangles be placed so that their sides CA, AD be in one straight line; wherefore[a] also EA and AB are in one straight line;* and join BD. Because the triangle ABC is equal to the triangle ADE, and that ABD is another triangle; therefore as the triangle CAB is to the triangle BAD, so is the triangle EAD to the triangle DAB:* but as the triangle CAB to the triangle BAD, so is the base CA to the base AD;* and as the triangle EAD to the triangle DAB, so is the base EA to the base AB;* therefore as CA to AD, so is EA to AB;* wherefore the sides of the triangles ABC, ADE about the equal angles are reciprocally proportional. * 14. 1. * 7. 5. * 1. 6. * 1. 6. * 11. 5.



Next, let the sides of the triangles ABC, ADE about

[a] *See the note to the last Proposition.*

the equal angles be reciprocally proportional, viz. CA to AD, as EA to AB; the triangle ABC shall be equal to the triangle ADE.

* Hyp. Join BD as before; then because,* as CA to AD, so is EA to AB; and as CA to AD, so is the triangle
* 1. 6. ABC to the triangle BAD;* and as EA to AB, so is
* 1. 6. the triangle EAD to the triangle BAD;* therefore as the triangle BAC to the triangle BAD, so is the
* 11. 5. triangle EAD to the triangle BAD;* that is, the triangles BAC, EAD have the same ratio to the triangle
* 9. 5. BAD: wherefore the triangle ABC is equal* to the triangle ADE. Therefore, equal triangles, &c. Q. E. D.

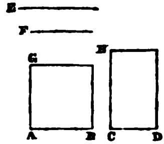
PROP. XVI. THEOR.

If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means.

And if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.

Let the four straight lines, AB, CD, E, F be proportionals, viz. as AB to CD so E to F; the rectangle contained by AB, F, shall be equal to the rectangle contained by CD, E.

* 11. 1. From the points A, C draw* AG, CH at right
* 3. 1. angles to AB, CD; and make* AG equal to F, and CH
* 31. 1. equal to E, and complete* the parallelograms BG, DH. Because, as AB to CD, so is E to F; and that E is
* 7. 5. equal to CH, and F to AG; therefore* AB is to CD, as CH to AG: that is, the sides of the parallelograms BG, DH about the equal angles are reciprocally proportional; but parallelograms which have their sides about equal angles reciprocally proportional, are equal
* 14. 6. to one another;* therefore the parallelogram BG is equal to the parallelogram DH: and the parallelogram



BG is contained by the straight lines AB, F; because AG is equal to F; and the parallelogram DH is contained by CD and E; because CH is equal to E: therefore the rectangle contained by the straight lines AB, F, is equal to that which is contained by CD and E.

Next, let the rectangle contained by the straight lines AB, F, be equal to that which is contained by CD, E; these four lines shall be proportionals, viz. AB shall be to CD, as E to F.

The same construction being made, because the rectangle contained by the straight lines AB, F, is equal to that which is contained by CD, E, and that the rectangle BG is contained by AB, F; because AG is equal to F; and the rectangle DH by CD, E, because CH is equal to E; therefore the parallelogram BG is equal* to the parallelogram DH; and they are equiangular: but the sides about the equal angles of equal parallelograms are reciprocally proportional;* wherefore, as AB to CD, so is CH to AG; but CH is equal to E, and AG to F: therefore* as AB is to CD, so E to F. Wherefore, if four, &c. Q. E. D.

* 1 Ax.
* 14. 6.
* 7. 5.

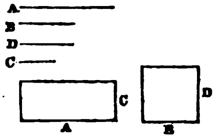
PROP. XVII. THEOR.

If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square of the mean.

And if the rectangle contained by the extremes be equal to the square of the mean, the three straight lines are proportionals.

Let the three straight lines A, B, C be proportionals, viz. as *A to B, so B to C;* the rectangle contained by *A, C, shall be equal to* the square of B.

Take D equal to B; and because as A to B, so B to
* 7. 5. C, and that B is equal to D; A is to B, as D to C:*
but if four straight lines be proportionals, the rectangle contained by the extremes is equal to that which is con-
* 16. 6. tained by the means:* therefore the rectangle contained by A, C, is equal to that contained by B, D: but the rectangle contained by B, D, is the square of B; because B is equal to D: therefore the rectangle contained by A, C, is equal to the square of B.



Next, let the rectangle contained by A, C, be equal to the square of B; A shall be to B, as B to C.

The same construction being made, because the rectangle contained by A, C, is equal to the square of B, and the square of B is equal to the rectangle contained by B, D, because B is equal to D; therefore the rectangle contained by A, C, is equal to that contained by B, D: but if the rectangle contained by the extremes be equal to that contained by the means, the
* 16. 6. four straight lines are proportionals;* therefore A is to B, as D to C; but B is equal to D; wherefore as A to B, so B to C. Therefore, if three straight lines, &c. Q. E. D.

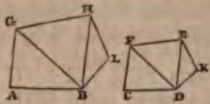
PROP. XVIII. PROB.

Upon a given straight line to describe a rectilineal figure, similar, and similarly situated, to a given rectilineal figure.

Let AB be the given straight line, and CDEF the given rectilineal figure of four sides; it is required upon the given straight line AB to describe a rectilineal figure, similar, and similarly situated, to CDEF.

Join DF, and at the points A, B, in the straight line

AB, make* the angle BAG equal to the angle at C, *23. 1.
and the angle ABG equal to the angle CDF; therefore the remaining angle AGB is equal to the remaining angle CFD:* wherefore the triangle GAB is equiangular to the triangle FCD: again, at the points G, B, in the straight line GB, make* the angle BGH equal to the angle DFE, and the angle GBH equal to FDE; therefore the remaining angle GHB is equal to the remaining angle FED, and the triangle GBH equiangular to the triangle FDE: then, because the angle AGB is equal to the angle CFD, and BGH to DFE, the whole angle AGH is equal* to the whole CFE: for the same reason, the angle ABH is equal to the angle CDE; also the angle at A is* equal to the angle at C, and the angle GHB to FED: therefore the rectilineal figure ABHG is equiangular to CDEF: also these figures have their sides about the equal angles proportionals: because the triangles GAB, FCD are equiangular, BA is* to AG, as DC to CF; and AG is to GB, as CF to FD; also the triangles BGH, DFE being equiangular, GB is to GH, as FD to FE; therefore, ex æquali,* AG is to GH, as CF to FE. In the same manner it may be proved that AB is to BH, as CD to DE: and GH is to HB, as FE to ED:* wherefore, because the rectilineal figures ABHG, CDEF are equiangular, and have their sides about the equal angles proportionals, they are similar* to one another.



{*32. 1. *3Ax.}
*23. 1.
*2 Ax.
*Const.
*4. 6.
*22. 5.
*4. 6.
*1 Def. 6.

Next, let it be required to describe upon a given straight line AB, a rectilineal figure, similar, and similarly situated, to the rectilineal figure CDKEF.

Join DE, and upon the given straight line AB describe the rectilineal figure ABHG similar and similarly situated, to the quadrilateral figure CDEF, by the

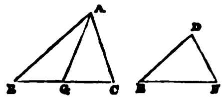
former case; and at the points B, H in the straight line BH, make the angle HBL equal to the angle EDK, and the angle BHL equal to the angle DEK;
* 32. 1. * 3 Ax. therefore the remaining angle at L is equal* to the remaining angle at K: and because the figures ABHG,
* 1 Def. 6. CDEF are similar, the angle GHB is equal* to the angle FED, and BHL is equal to DEK; wherefore the whole angle GHL is equal to the whole angle FEK: for the same reason the angle ABL is equal to the angle CDK: therefore the five sided figures AGHLB, CFEKD are equiangular; and because the
* 1 Def. 6. figures AGHB, CFED are similar,* GH is to HB, as
* 4. 6. FE to ED; but as HB to HL, so is ED to EK;* there-
* 22. 5. fore, ex æquali,* GH is to HL, as FE to EK: for the same reason, AB is to BL as CD to DK: and BL is
* 4. 6. to LH, as* DK to KE, because the triangles BLH, DKE are equiangular: therefore, because the five-sided figures AGHLB, CFEKD are equiangular, and have their sides about the equal angles proportionals, they are similar to one another. In the same manner upon a given straight line, a rectilineal figure may be described, similar, and similarly situated, to a given rectilineal figure of six, or any number of sides. Which was to be done.

PROP. XIX. THEOR.

Similar triangles are to one another in the duplicate ratio of their homologous sides.

Let ABC, DEF be similar triangles, having the angle B equal to the angle E, and let AB be to BC, as
* 12 Def. 5. DE to EF, so that the side BC may be* homologous to EF: the triangle ABC shall have to the triangle DEF, the duplicate ratio of that which BC has to EF.
* 11. 6. Take* BG a third proportional to BC, EF, so that BC may be to EF, as EF to BG, and join GA: then,

because as AB to BC, so DE to EF, alternately,* AB * 16. 5.
is to DE,* as BC to EF: but* as BC to EF, so is EF to * Const.
BG; therefore* as AB to DE, so is EF to BG: there- * 11. 5.
fore the sides of the triangles ABG, DEF which are about the equal angles, are reciprocally proportional: but triangles which have the sides about two equal angles reciprocally proportional are equal* to one * 15. 6.
another: therefore the triangle ABG is equal to the triangle DEF: and because



as BC is to EF, so EF to BG; and that if three straight lines be proportional, the first is said* to have * 10 Def. 5.
to the third the duplicate ratio of that which it has to the second; therefore BC has to BG the duplicate ratio of that which BC has to EF: but as BC to BG, so is* the triangle ABC to the triangle ABG: there- * 1. 6.
fore the triangle ABC has to the triangle ABG the duplicate ratio of that which BC has to EF: but the triangle ABG is equal to the triangle DEF; therefore the triangle ABC has to the triangle DEF the duplicate ratio of that which BC has to EF. Therefore similar triangles, &c. Q. E. D.

Cor. From this it is manifest, that if three straight lines be proportional, as the first is to the third, so is any triangle upon the first, to a similar and similarly described triangle upon the second.

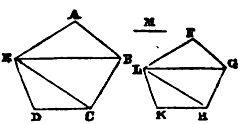
PROP. XX. THEOR.

Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous sides have.

Let ABCDE, FGHKL be similar polygons, and let

AB be the side homologous to FG: the polygons ABCDE, FGHKL may be divided into the same number of similar triangles, whereof each shall have to each the same ratio which the polygons have; and the polygon ABCDE shall have to the polygon FGHKL the duplicate ratio of that which the side AB has to the side FG.

Join BE, EC, GL, LH: and because the polygon ABCDE is similar to the polygon FGHKL, the angle
* 1 Def. 6. BAE is equal* to the angle GFL, and BA is to AE,
* 1 Def. 6. as GF to FL:* therefore because the triangles ABE, FGL have an angle in the one equal to an angle in the other, and their sides about these equal angles pro-
* 6. 6. portionals, the triangle ABE is equiangular* to the
* 4. 6. triangle FGL, and therefore* similar to it; wherefore the angle ABE is equal to the angle FGL: and, because the polygons are similar, the whole angle ABC
* 1 Def. 6. is equal* to the whole angle FGH; therefore the re-
* 3 Ax. maining angle EBC is equal* to the remaining angle LGH: and because the triangles ABE, FGL are
* 4. 6. similar, EB is to BA, as LG to GF;* and also, be-
* 1 Def. 6. cause the polygons are similar,* AB is to BC, as FG to
* 22. 5. GH; therefore, ex æquali,* EB is to BC, as LG to GH; that is, the sides about the equal angles EBC,
* 6. 6. LGH are proportionals; therefore* the triangle EBC
* 4. 6. is equiangular to the triangle LGH, and similar* to it:



for the same reason, the triangle ECD is similar to the triangle LHK: therefore the similar polygons ABCDE, FGHKL are divided into the same number of similar triangles.

Also these triangles shall have, each to each, the same ratio which the polygons have to one another, the antecedents being ABE, EBC, ECD, and the consequents FGL, LGH, LHK: and the polygon ABCDE shall have to the polygon FGHKL the duplicate ratio of that which the side AB has to the homologous side FG.

Because the triangle ABE is similar to the triangle FGL, ABE has to FGL the duplicate ratio * of that * 19. 6.
which the side BE has to the side GL: for the same reason, the triangle BEC has to GLH the duplicate ratio of that which BE has to GL: therefore, as the triangle ABE is to the triangle FGL, so * is the tri- * 11. 5.
angle BEC to the triangle GLH. Again, because the triangle EBC is similar to the triangle LGH, therefore EBC has to LGH the duplicate ratio of that which the side EC has to the side LH: for the same reason, the triangle ECD has to the triangle LHK, the duplicate ratio of that which EC has to LH: therefore as the triangle EBC is to the triangle LGH, so is * the tri- * 11. 5.
angle ECD to the triangle LHK: but it has been proved that the triangle EBC is to the triangle LGH, as the triangle ABE to the triangle FGL: therefore, as the triangle ABE is to the triangle FGL, so is the triangle EBC to the triangle LGH, and the triangle ECD to the triangle LHK: and therefore, as one of the antecedents is to one of the consequents,* so are all * 12. 5.
the antecedents to all the consequents: therefore, as the triangle ABE is to the triangle FGL, so is the polygon ABCDE to the polygon FGHKL: but the triangle ABE has to the triangle FGL, the duplicate ratio* of that which the side AB has to the homologous * 19. 6.
side FG; therefore the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which AB has to the homologous side FG. Wherefore, similar polygons, &c. Q. E. D.

COR. 1. In like manner, it may be proved, that similar four-sided figures, or of any number of sides, are one to another in the duplicate ratio of their
* 19. 6. homologous sides, and it has already been proved* in triangles: therefore, universally, similar rectilineal figures are to one another in the duplicate ratio of their homologous sides.

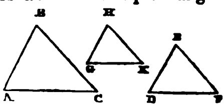
COR. 2. And if to AB, FG, two of the homologous
* 11. 6. sides, a third* proportional M be taken, AB has to M
* 10 Def. 5. the duplicate ratio* of that which AB has to FG: but the four-sided figure or polygon upon AB has to the four-sided figure or polygon upon FG, the duplicate
* Cor. 1. ratio* of that which AB has to FG: therefore, as AB is
* 11. 5. to M, so is* the figure upon AB to the figure upon FG,
* Cor. 19. 6. which was also proved in triangles:* therefore, universally, it is manifest, that if three straight lines be proportional, as the first is to the third, so is any rectilineal figure upon the first, to a similar and similarly described rectilineal figure upon the second.

PROP. XXI. THEOR.

Rectilineal figures which are similar to the same rectilineal figure, are also similar to one another.

Let each of the rectilineal figures ABC, DEF be similar to the rectilineal figure GHK: the figure ABC shall be similar to the figure DEF.

Because ABC is similar to GHK, they are equiangular, and have their sides about the equal angles
* 1 Def. 6. proportionals.* Again, because DEF is similar to GHK, they are equiangular, and have their sides about



* 1 Def. 6. the equal angles proportionals:* therefore the figures ABC and DEF are each of them equiangular to GHK,

and have the sides about the equal angles of each of them and of GHK proportionals: therefore the rectilineal figures ABC and DEF are equiangular,* and have their sides about the equal angles proportionals:* therefore ABC is similar* to DEF. Therefore, rectilineal figures, &c. Q. E. D.

* 1 Ax. 1.
* 11. 5.
* 1 Def. 6.

PROP. XXII. THEOR.

If four straight lines be proportionals, the similar rectilineal figures similarly described upon them shall also be proportionals; and if the similar rectilineal figures similarly described upon four straight lines be proportionals, those straight lines shall be proportionals.

Let the four straight lines AB, CD, EF, GH be proportionals, viz. AB to CD, as EF to GH, and upon AB, CD let the similar rectilineal figures KAB, LCD be similarly described; and upon EF, GH the similar rectilineal figures MF, NH, in like manner: the rectilineal figure KAB shall be to LCD, as MF to NH.

To AB, CD take a third proportional* X; and to EF, GH a third proportional O: and because AB is to CD, as EF to GH, and that CD is* to X, as GH to O; therefore, ex æquali,* as AB to X, so EF to O: but as AB to X, so is* the rectilineal KAB to the rectilineal figure LCD, and as EF to O, so is* the rectilineal figure MF to the rectilineal figure NH: therefore, as KAB to LCD, so* is MF to NH.

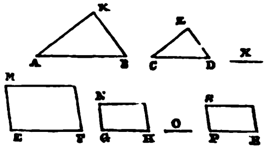
* 11. 6.
* 11. 5.
* 22. 5.
* 2 Cor. 20. 6.
* 2 Cor. 20. 6.
* 11. 5.

Next, let the rectilineal figure KAB be to LCD, as MF to NH; the straight line AB shall be to CD, as EF to GH.

Make* as AB to CD, so EF to PR, and upon PR describe* the rectilineal figure SR similar and similarly situated to either of the figures MF, NH: then, because as AB to CD, so is EF to PR, and that upon

* 12. 6.
* 18. 6.

AB, CD are described the similar and similarly situated rectilineals KAB, LCD, and upon EF, PR, in like manner, the similar rectilineals MF, SR; therefore KAB is to LCD, as MF to SR; but, by the hypothesis, KAB is to LCD, as MF to NH; and therefore the



rectilineal figure MF has the same ratio to each of the
* 9. 5. two NH, SR: therefore NH is equal* to SR; and they are similar, and similarly situated; therefore GH is equal to PR: and because as AB to CD, so is EF to
* 7. 5. PR, and that PR is equal to GH; therefore* AB is to CD, as EF to GH. If, therefore, four straight lines, &c. Q. E. D.

PROP. XXIII. THEOR.

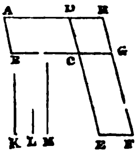
Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.

Let AC, CF be equiangular parallelograms, having the angle BCD equal to the angle ECG: the parallelogram AC shall have to the parallelogram CF, the ratio which is compounded of the ratios of their sides.

Let BC, CG be placed in a straight line; therefore DC and CE are also in a straight line;[a] and complete the parallelogram DG; and, taking any straight line
* 12. 6. K, make* as BC to CG, so K to L; and as DC to
* 12. 6. CE, so make* L to M: therefore the ratios of K to L, and L to M, are the same with the ratios of the sides,

[a] See the proof of this in the note to Prop. 14. 6.

viz. of BC to CG, and DC to CE: but the ratio of K to M is that which is said to be compounded* of the ratios of K to L, and L to M: therefore K has to M the ratio compounded of the ratios of the sides: but because as BC to CG, so is the parallelogram AC to the parallelogram CH;* and as BC to CG, so is K to L; therefore K is* to L, as the parallelogram AC to the parallelogram CH. Again, because as DC to CE, so is the parallelogram CH to the parallelogram CF; and as DC to CE, so is L to M; therefore* L is to M, as the parallelogram CH to the parallelogram CF: but since it has been proved, that as K to L, so is the parallelogram AC to the parallelogram CH; and as L to M, so is the parallelogram CH to the parallelogram CF; therefore, ex æquali,* K is to M, as the parallelogram AC to the parallelogram CF: but K has to M the ratio which is compounded of the ratios of the sides; therefore the parallelogram AC has to the parallelogram CF the ratio which is compounded of the ratios of the sides. Wherefore equiangular parallelograms, &c. Q. E. D.



* A. Def. 5.
* 1. 6.
* 11. 5.
* 11. 5.
* 22. 5.

PROP. XXIV. THEOR.

The parallelograms about the diameter of any parallelogram, are similar to the whole, and to one another.

Let ABCD be a parallelogram, of which the diameter is AC; and EG, HK the parallelograms about the diameter: each of the parallelograms EG, HK shall be similar to the whole parallelogram ABCD, and they shall also be similar to one another.

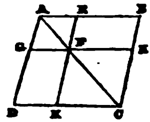
Because DC, GF are parallels, the angle ADC is equal* to the angle AGF: and because BC, EF are parallels, the angle ABC is equal to the angle AEF:

* 29. 1.

* 34. 1. and each of the angles BCD, EFG is equal* to the opposite angle DAB, and therefore they are equal to one another; therefore the parallelograms ABCD, AEFG are equiangular. And because the angle ABC is equal to the angle AEF, and the angle BAC common to the two triangles BAC, EAF, they are equiangular

* 4. 6. to one another; therefore* as AB to

* 34. 1. BC, so is AE to EF: and because* the opposite sides of parallelograms



* 7. 5. are equal to one another, AB is* to AD as AE to AG; and DC to CB as GF to FE: and also CD to DA as FG to GA: therefore the sides of the parallelograms ABCD, AEFG about the equal angles are proportionals; and they are

* 1 Def. 6. therefore similar* to one another: for the same reason, the parallelogram ABCD is similar to the parallelogram FHCK: therefore each of the parallelograms GE, KH is similar to DB: but rectilineal figures which are similar to the same rectilineal figure, are

* 21. 6. similar* to one another: therefore the parallelogram GE is similar to KH. Wherefore the parallelograms, &c. Q. E. D.

PROP. XXV. PROB.

To describe a rectilineal figure which shall be similar to one, and equal to another given rectilineal figure.

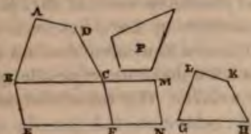
Let ABCD be the given rectilineal figure, to which the figure to be described is required to be similar, and P that to which it must be equal: it is required to describe a rectilineal figure similar to ABCD, and equal to P.

* Cor. 45. 1. Upon the straight line BC describe* the parallelogram BF equal to the figure ABCD; and upon CF

* Cor. 45. 1. describe* the parallelogram CN equal to P, and having the angle FCM equal to the angle CBE: therefore

BC and CM are in a straight line,* as also EF and FN:[a] between BC and CM find* a mean proportional GH, and upon GH describe* the rectilineal figure GHKL, similar and similarly situated to the figure ABCD: then shall the rectilineal figure GHKL be similar to BCDA, and equal to P. {* 29. 1. * 14. 1.} * 13. 6. * 18. 6.

Because BC is to GH as GH to CM, and if three straight lines be proportionals, as the first is to the third, so is* the figure upon the first to the similar and similarly described figure upon the second; therefore as BC to CM, so is the rectilineal figure ABCD to the rectilineal figure GHKL: but as BC to CM, so is* the parallelogram BF to the parallelogram CN: therefore as the rectilineal figure ABCD is to the rectilineal figure GHKL, so is the parallelogram BF to the parallelogram CN:* but the rectilineal figure * 2 Cor. 20. 6. * 1. 6. * 11. 5.



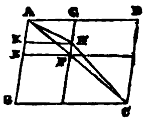
ABCD is equal* to the parallelogram BF; therefore the rectilineal figure GHKL is equal* to the parallelogram CN: but CN is equal* to the figure P; wherefore GHKL is equal to P; and it is similar to ABCD; therefore the rectilineal figure GHKL has been described similar to the figure ABCD, and equal to P. Which was to be done. * Const. * 14. 5. * Const.

[a] Because the opposite angles of parallelograms are equal to one another,* the two angles CFE, CFN are equal to the two angles CBE, CMN; but the two angles CBE, CMN are equal to two right angles,* because MN, BE are each of them parallel to CF, and therefore they are parallel to one another; hence the angles CFE, CFN are equal to two right angles, and therefore* EF, FN *are in the same straight line.* * 34. 1. * 29. 1. * 14. 1.

PROP. XXVI. THEOR.

If two similar parallelograms have a common angle, and be similarly situated; they are about the same diameter.

Let the parallelograms ABCD, AEFG be similar and similarly situated, and have the angle DAB common: then ABCD and AEFG shall be about the same diameter.



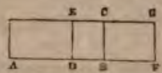
For, if not, let, if possible, the parallelogram BD have its diameter AHC in a different straight line from AF the diameter of the parallelogram EG, and let GF meet AHC in H; and through H draw HK parallel to AD or BC.

Because the parallelograms ABCD, AKHG are about
* 24. 6. the same diameter, they are similar* to one another:
* 1 Def. 6. therefore as DA to AB, so is* GA to AK: but because
* Hyp. ABCD and AEFG are similar* parallelograms, as DA
* 11. 5. is to AB, so is GA to AE; therefore* as GA to AE, so GA to AK; wherefore GA has the same ratio to each of the straight lines AE, AK; and consequently
* 9. 5. AK is equal* to AE, the less equal to the greater, which is impossible: therefore ABCD and AKHG are not about the same diameter: therefore ABCD and AEFG must be about the same diameter. Therefore, if two similar, &c. Q. E. D.

'To understand the three following propositions 'more easily, it is to be observed,

1. 'That a parallelogram is said to be applied to a 'straight line, when it is described upon it as one of 'its sides. Ex. gr. the parallelogram AC is said to be 'applied to the straight line AB.

2. 'But a parallelogram AE is said to be applied to 'a straight line AB, deficient by a parallelogram, when



'AD the base of AE is less than 'AB, and therefore AE is less 'than the parallelogram AC de-'scribed upon AB in the same 'angle, and between the same parallels, by the paral-'lelogram DC; and DC is therefore called the defect 'of AE.

3. 'And a parallelogram AG is said to be applied 'to a straight line AB, exceeding by a parallelogram, 'when AF the base of AG is greater than AB, and 'therefore AG exceeds AC the parallelogram described 'upon AB in the same angle, and between the same 'parallels, by the parallelogram BG.'

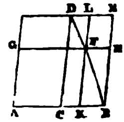
PROP. XXVII. THEOR.

Of all parallelograms applied to the same straight line, and deficient by parallelograms, similar and similarly situated to that which is described upon the half of the line; that which is applied to the half, and is similar to its defect, is the greatest.

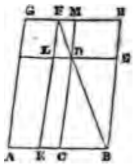
Let AB be a straight line divided into two equal parts in C, and let the parallelogram AD be applied to the half AC, which is therefore deficient from the parallelogram upon the whole line AB by the parallelogram CE upon the other half CB: of all the parallelograms applied to any other parts of AB, and deficient by parallelograms that are similar and similarly situated to CE, AD shall be the greatest.

Let AF be any parallelogram applied to AK, any other part of AB than the half, so as to be deficient from the parallelogram upon the whole line AB by the parallelogram KH, similar and similarly situated to CE; AD shall be greater than AF.

First, let AK the base of AF, be greater than AC the half of AB; and because CE is similar* to the *Hyp.



parallelogram KH, they are about the
* 26. 6. same diameter.* Draw their diameter
DB, and complete the scheme: then,
* 43. 1. because the parallelogram CF is equal*
to FE, add KH to both, therefore the
* 36 1. whole CH is equal to the whole KE: but CH is equal*
to CG, because the base AC is equal to the base CB;
* 1 Ax therefore CG is equal* to KE: to each of these equals
* 2 Ax. add CF; then the whole AF is equal* to the gnomon
CHL: therefore CE, or the parallelogram AD, is
greater than the parallelogram AF.



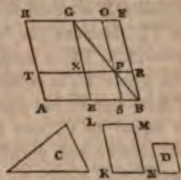
Next, let AK the base of AF, be
less than AC; the same construction
being made, because BC is equal CA,
* 34. 1. therefore HM is equal* to MG; where-
* 36. 1. fore the parallelogram DH is equal*
to DG; therefore DH is greater
* 43. 1. than LG: but DH is equal* to DK;
therefore DK is greater than LG: to each of these add
AL; then the whole AD is greater than the whole
AF. Therefore, of all parallelograms applied, &c.
Q. E. D.

PROP. XXVIII. PROB.

To a given straight line to apply a parallelogram equal to a given rectilineal figure, and deficient by a parallelogram similar to a given parallelogram: but the given rectilineal figure to which the parallelogram to be applied is to be equal, must not be greater than the parallelogram applied to half of the given line, having its defect similar to the defect of that which is to be applied; that is, to the given parallelogram.

Let AB be the given straight line, and C the given rectilineal figure, to which the parallelogram to be applied is required to be equal, which figure must not

be greater* than the parallelogram applied to the half * 27. 6.
of the line having its defect from that upon the whole line similar to the defect of that which is to be applied; and let D be the parallelogram to which this defect is required to be similar. It is required to apply a parallelogram to the straight line AB, which shall be equal to the figure C, and be deficient from the parallelogram upon the whole line by a parallelogram similar to D.



Divide AB into two equal parts* in the point E, and upon * 10. 1.
EB describe the parallelogram EBFG similar* and * 18. 6.
similarly situated to D, and complete the parallelogram AG, which must either be equal to C, or greater than it, by the determination.

If AG be equal to C, then what was required is already done: for, upon the straight line AB, the parallelogram AG is applied equal to the figure C, and deficient by the parallelogram EF similar to D. But, if AG be not equal to C, it is greater than it; and EF is equal* to AG; therefore EF is also greater * 36. 1.
than C. Make* the parallelogram KLMN equal to * 25. 6.
the excess of EF above C, and similar and similarly situated to D; but, by construction, D is similar to EF, therefore* also KM is similar to EF. Let KL * 21. 6.
be the side homologous to EG, and LM to GF: then, because EF is equal to C and KM together, EF is greater than KM; therefore the straight line EG is greater than KL, and GF than LM: make* GX equal * 3. 1.
to LK, and GO equal to LM, and complete* the paral- * 31. 1.
lelogram XGOP: therefore XO is equal and similar to KM; but KM is similar to EF; wherefore XO is also similar to EF, and therefore* XO and EF are about * 26. 6.
the same diameter: let GPB be their diameter and

complete the scheme. Then, because EF is equal to C and KM together, and XO a part of the one, is equal to KM a part of the other, the remainder, viz. the
* 3 Ax. gnomon ERO, is equal* to the remainder C: and be-
* 43. 1. cause OR is equal* to XS, by adding SR to each, the
* 36. 1. whole OB is equal to the whole XB: but XB is equal* to TE, because the base AE is equal to the base EB;
* 1 Ax. wherefore TE is equal* to OB: add XS to each, then the whole TS is equal to the whole, viz. to the gnomon ERO: but it has been proved that the gnomon ERO is equal to C, and therefore TS is also equal to C. Wherefore the parallelogram TS, equal to the given rectilineal figure C, is applied to the given straight line AB, deficient by the parallelogram SR, similar
* 24. 6. to the given one D, because SR is similar* to EF. Which was to be done.

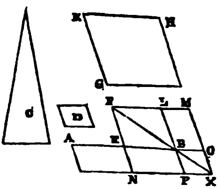
PROP. XXIX. PROB.

To a given straight line to apply a parallelogram equal to a given rectilineal figure, exceeding by a parallelogram similar to another given.

Let AB be the given straight line, and C the given rectilineal figure to which the parallelogram to be applied is required to be equal, and D the parallelogram to which the excess of the one to be applied above that upon the given line is required to be similar. It is required to apply a parallelogram to the given straight line AB which shall be equal to the figure C, exceeding by a parallelogram similar to D.

* 10. 1. Divide* AB into two equal parts in the point E, and
* 18. 6. upon EB describe* the parallelogram EL similar, and
* 25. 6. similarly situated to D: and make* the parallelogram GH equal to EL and C together, and similar and simi-
* 21. 6. larly situated to D; wherefore GH is similar* to EL: let KH be the side homologous to FL, and KG to FE:

and because the parallelogram GH is greater than EL, therefore the side KH is greater than FL, and KG than FE. Produce FL and FE, and make FLM equal to KH, and FEN to KG, and complete the parallelogram MN: therefore MN is equal and similar to GH; but GH is similar to EL; wherefore MN is similar to EL, and consequently EL and MN are about the same diameter.* Draw their diameter FX, and complete the scheme. Then, because GH is equal to EL and C together, and that GH is equal to MN; therefore MN is equal to EL and C: take away the common part EL; and the remainder, viz. the gnomon NOL, is equal to C. And because AE is equal to EB, the parallelogram AN is equal* to the parallelogram NB, that is, to BM:* add NO to each; therefore the whole, viz. the parallelogram AX, is equal to the gnomon NOL: but the gnomon NOL is equal to C; therefore also AX is equal to C. Wherefore to the straight line AB, there is applied the parallelogram AX equal to the given rectilineal C, exceeding by the parallelogram PO, which is similar to D, because PO is similar* to EL. Which was to be done.



* 26. 6.
* 36. 1.
* 43. 1.
* 24. 6.

PROP. XXX. PROB.

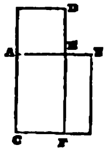
To cut a given straight line in extreme and mean ratio.

Let AB be the given straight line; it is required to cut it in extreme and mean ratio.

Upon AB describe* the square BC, and to AC apply the parallelogram CD equal to BC, exceeding by the figure AD similar to BC:* then shall AB be cut in extreme and mean ratio in E. Since BC is a square,

* 46. 1.
* 29. 6.

AD is also a square; and because BC is equal to CD, by taking the common part CE from each, the remainder BF is equal to the remainder AD: and these figures are equiangular; therefore their sides about the equal angles are recipro-



* 14. 6. cally* proportional: wherefore, as FE
* 34. 1. to ED, so AE to EB: but FE is equal* to AC, that is, to AB; and ED is equal to AE: therefore as BA to AE, so is AE to EB: but AB is greater than AE;
* 14. 5. wherefore AE is greater* than EB: therefore the straight line AB is cut in extreme and mean ratio in
* 3 Def. 6. E.* Which was to be done.

Otherwise:

Let AB be the given straight line; it is required to cut it in extreme and mean ratio.

* 11. 2. Divide* AB in the point C, so that the rectangle contained by AB, BC may be equal to the square of AC: then shall AB be cut in extreme and mean ratio in C.

Because the rectangle AB, BC is equal to the square
* 17. 6. of AC, as BA to AC, so is* AC to CB: therefore AB
* 3 Def. 6. is cut in extreme and mean ratio in C.* Which was to be done.

PROP. XXXI. THEOR.

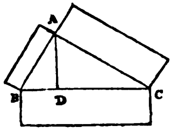
In right-angled triangles, the rectilineal figure described upon the side opposite to the right angle, is equal to the similar and similarly described figures upon the sides containing the right angle.

Let ABC be a right-angled triangle, having the right angle BAC: the rectilineal figure described upon BC shall be equal to the similar and similarly described figures upon BA, AC.

* 12. 1. Draw* the perpendicular AD; therefore, because in

the right-angled triangle ABC, AD is drawn from the right angle at A, perpendicular to the base BC, the triangles ABD, ADC are similar* to the whole triangle ABC, and to one another, and because the triangle ABC is similar to ADB, as CB to BA, so is* BA to BD; and because these three straight lines are proportionals, as the first is to the third, so is the figure upon the first to the similar and similarly described figure* upon the second: therefore as CB to BD, so is the figure upon CB to the similar and similarly described figure upon BA: and, inversely,* as DB to BC, so is the figure upon BA to that upon BC: for the same reason, as DC to CB, so is the figure upon CA to that upon CB; wherefore as BD and DC together to BC, so are* the figures upon BA, AC to that upon BC: but BD and DC together are equal to BC: therefore the figure described on BC is equal* to the similar and similarly described figures on BA, AC. Wherefore, in right-angled triangles, &c. Q. E. D.

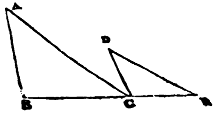
* 8. 6.
* 4. 6.
* 2 Cor. 20. 6.
* B. 5.
* 24. 5.
* A. 5.



PROP. XXXII. THEOR.

If two triangles which have two sides of the one proportional to two sides of the other, be joined at one angle, so as to have their homologous sides parallel to one another; the remaining sides shall be in a straight line.

Let ABC, DCE be two triangles which have the two sides BA, AC proportional to the two CD, DE, viz. BA to AC, as CD to DE; and let AB be parallel to DC, and AC to DE, BC and CE shall be in a straight line.



Because AB is parallel to

DC, and the straight line AC meets them, the alter-
* 29. 1. nate angles* BAC, ACD are equal; for a like reason, the angle CDE is equal to the angle ACD; wherefore
* 1 Ax. also BAC is equal* to CDE: and because the triangles ABC, DCE have the angle A equal to the angle D, and the sides about these equal angles proportionals, viz. BA to AC, as CD to DE, the triangle ABC is
* 6. 6 equiangular* to DCE: therefore the angle ABC is equal to the angle DCE: and the angle BAC was proved to be equal to ACD: therefore the whole angle
* 2 Ax. ACE is equal* to the two angles ABC, BAC; add to each of these equals the common angle ACB, then the angles ACE, ACB are equal to the angles ABC, BAC,
* 32. 1. ACB: but ABC, BAC, ACB are equal* to two right angles; therefore also the angles ACE, ACB are equal to two right angles; and since at the point C, in the straight line AC, the two straight lines BC, CE, which are on the opposite sides of it, make the adjacent angles ACE, ACB equal to two right angles; there-
* 14. 1. fore* BC and CE are in a straight line. Wherefore, if two triangles, &c. Q. E. D.

PROP. XXXIII. THEOR.

In equal circles, angles, whether at the centres or circumferences, have the same ratio which the circumferences on which they stand have to one another: so also have the sectors.

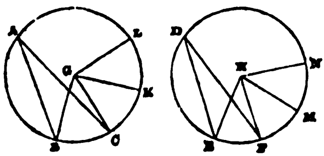
Let ABC, DEF be equal circles; BGC, EHF angles at their centres, and BAC, EDF angles at their circumferences; then the circumference BC shall be to the circumference EF, as the angle BGC to the angle EHF, and the angle BAC to the angle EDF; and also the sector BGC to the sector EHF.

Take any number of circumferences CK, KL, each equal to BC, and any number whatever FM, MN each

equal to EF: and join GK, GL, HM, HN. Because the circumferences BC, CK, KL are all equal, the angles BGC, CGK, KGL are also all* equal: therefore what multiple soever the circumference BL is of the circumference BC, the same multiple is the angle BGL of the angle BGC: for the same reason, whatever multiple the circumference EN is of the circumference EF, the same multiple is the angle EHN of the angle EHF: and if the circumference BL be equal to the circumference EN, the angle BGL is also equal* to the angle EHN; and if the circumference BL be greater than EN, the angle BGL is greater than EHN; and if less, less: therefore since there are four magnitudes, the two circumferences BC, EF, and the two angles BGC, EHF; and that of the circumference BC, and of the angle BGC, have been taken any equimultiples whatever, viz. the circumference BL, and the angle BGL; and of the circumference EF, and of the angle EHF, any equimultiples whatever, viz. the

* 27. 3.

* 27. 3.



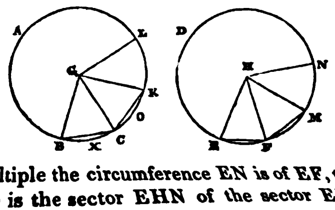
circumference EN, and the angle EHN: and it has been proved, that, if the circumference BL be greater than EN, the angle BGL is greater than EHN; and if equal, equal; and if less, less: therefore as the circumference BC to the circumference EF, so* is the angle BGC to the angle EHF: but as the angle BGC is to the angle EHF, so is* the angle BAC to the angle EDF, for each is double* of each: therefore, as the circumference *BC is to EF, so is* the angle BGC to the *angle EHF, and the angle* BAC to the angle EDF.

* 5 Def. 5.

* 15. 5.

* 20. 3.

Also, as the circumference BC to EF, so shall the sector BGC be to the sector EHF. Join BC, CK, and in the circumferences BC, CK take any points X, O, and join BX, XC, CO, OK: then, because in the triangles GBC, GCK the two sides BG, GC are equal to the two CG, GK, and that they contain equal angles;
* 4. 1. the base BC is equal* to the base CK, and the triangle GBC to the triangle GCK: and because the circumference BC is equal to the circumference CK, the remaining part of the whole circumference of the circle
* 3 Ax. ABC, is equal* to the remaining part of the whole circumference of the same circle: wherefore the angle
* 27. 3. BXC is equal* to the angle COK; and the segment
* 11 Def. 3. BXC is therefore similar* to the segment COK; and they are upon equal straight lines BC, CK: but similar segments of circles upon equal straight lines,
* 24. 3. are equal* to one another: therefore the segment BXC is equal to the segment COK: and the triangle BGC was proved equal to the triangle CGK; therefore the whole, the sector BGC, is equal to the whole, the sector CGK: for the same reason, the sector KGL is equal to each of the sectors BGC, CGK. In the same manner, the sectors EHF, FHM, MHN may be proved equal to one another: therefore, what multiple soever the circumference BL is of the circumference BC, the same multiple is the sector BGL of the sector BGC: for the same reason, what-



ever multiple the circumference EN is of EF, the same multiple is the sector EHN of the sector EHF: and

if the circumference BL be equal to EN, the sector BGL is equal to the sector EHN; and if the circumference BL be greater than EN, the sector BGL is greater than the sector EHN; and if less, less. Since, then, there are four magnitudes, the two circumferences BC, EF, and the two sectors BGC, EHF, and that of the circumference BC, and sector BGC, the circumference BL and sector BGL are any equal multiples whatever; and of the circumference EF, and sector EHF, the circumference EN and sector EHN, are any equimultiples whatever; and it has been proved, that if the circumference BL be greater than EN, the sector BGL is greater than the sector EHN; and if equal, equal; and if less, less: therefore,* as the circumference BC is to the circumference EF, so is the sector BGC to the sector EHF. Wherefore, in equal circles, &c. Q. E. D. * 5 Def. 5.

PROP. B. THEOR.

If an angle of a triangle be bisected by a straight line, which likewise cuts the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the straight line bisecting the angle.

Let ABC be a triangle, and let the angle BAC be bisected by the straight line AD; the rectangle BA, AC shall be equal to the rectangle BD, DC, together with the square of AD.

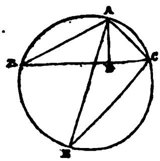
Describe* the circle ACB about the triangle, and produce AD to the circumference in E, and join EC: then because the angle BAD is equal* to the angle CAE, and the angle ABD to the angle* AEC, for they are in the same * 5. 4. * Hyp. * 21. 3.

* 32. 1. segment; the triangles ABD, AEC are equiangular* to
* 4. 6. one another: therefore as BA to AD, so is* EA to AC,
* 16. 6. and consequently the rectangle BA, AC is equal* to
* 3. 2. the rectangle EA, AD, that is,* to the rectangle ED, DA, together with the square of AD: but the rec-
* 35. 3. tangle ED, DA is equal* to the rectangle BD, DC: therefore the rectangle BA, AC is equal to the rectangle BD, DC, together with the square of AD. Wherefore, if an angle, &c. Q. E. D.

PROP. C. THEOR.

If from an angle of a triangle a straight line be drawn perpendicular to the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

Let ABC be a triangle, and AD the perpendicular from the angle A to the base BC; the rectangle BA, AC shall be equal to the rectangle contained by AD and the diameter of the circle described about the triangle.



* 5. 4. Describe* the circle ACB about the triangle, and draw the diameter AE, and join EC. Because the
* 31. 3. right angle BDA is equal* to the angle ECA in a semicircle, and
* 21. 3. the angle ABD equal* to the angle AEC in the same segment; the
* 4. 6. triangles ABD, AEC are equiangular: therefore as* BA to AD, so is EA to AC: and consequently the rec-
* 16. 6. tangle BA, AC is equal* to the rectangle EA, AD. If, therefore, from an angle, &c. Q. E. D.

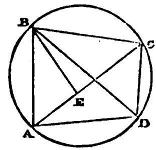
PROP. D. THEOR.

The rectangle contained by the diagonals of a quadri-

lateral inscribed in a circle, is equal to both the rectangles contained by its opposite sides.

Let ABCD be any quadrilateral inscribed in a circle, and join AC, BD; the rectangle contained by AC, BD shall be equal to the two rectangles contained by AB, CD, and by AD, BC.[a]

Make the angle ABE equal* to the angle DBC; add * 23. 1.
to each of these equals the common angle EBD, then the angle ABD is equal to the angle EBC: and the angle BDA is equal* to the angle BCE, because they * 21. 3.
are in the same segment; therefore the triangle ABD is equiangular to the triangle BCE: wherefore* * 4. 6.
as BC is to CE, so is BD to DA; and consequently the rectangle BC, AD is equal* to the rectangle BD, * 16. 6.
CE. Again, because the angle



ABE is equal to the angle DBC, and the angle* BAE * 21. 3.
to the angle BDC, the triangle ABE is equiangular to the triangle BCD: therefore as BA to AE, so is BD to DC; wherefore the rectangle BA, DC is equal to the rectangle BD, AE: but the rectangle BC, AD has been shown equal to the rectangle BD, CE; therefore[b] the whole rectangle AC, BD* is equal to the rectangle * 1. 2.
AB, DC, together with the rectangle AD, BC. Therefore, the rectangle, &c. Q. E. D.

[a] This is a Lemma of Cl. Ptolomæus, in page 9 of his Μεγαλη Συνταξις.

[b] The rectangles BC, AD and BA, DC are together equal* to * 2 Ax.
the rectangles BD, CE and BD, AE; that is* to the whole rec- * 1. 2.
tangle BD, AC.

THE

ELEMENTS OF EUCLID.

BOOK XI.

DEFINITIONS.

I.

A SOLID is that which hath length, breadth, and thickness.

II.

That which bounds a solid is a *superficies*.

III.

A straight line is *perpendicular*, or *at right angles* to a plane, when it makes right angles with every straight line meeting it in that plane.

IV.

A *plane is perpendicular to a plane*, when the straight lines drawn in one of the planes perpendicular to the common section of the two planes, are perpendicular to the other plane.

V.

The *inclination of a straight line to a plane* is the acute angle contained by that straight line, and another drawn from the point in which the first line meets the plane, to the point in which a perpendicular to the plane drawn from any point of the first line above the plane, meets the same plane.

VI.

The *inclination of a plane to a plane* is the acute angle contained by two straight lines drawn from any the same point of their common section at right angles to it, one upon one plane, and the other upon the other plane.

VII.

Two planes are said to have the same, or a like inclination to one another, which two other planes have, when the said angles of inclination are equal to one another.

VIII.

Parallel planes are such as do not meet one another though produced.

IX.

A *solid angle* is that which is made by the meeting, in one point, of more than two plane angles, which are not in the same plane.

X

The tenth definition is omitted for reasons given in the notes.' *See the Octavo Edition.*

XI.

Similar solid figures are such as have all their solid *angles equal, each to each,* and which are contained *by the same number* of similar planes.

XII.

A *pyramid* is a solid figure contained by planes that are constituted betwixt one plane and one point above it in which they meet.

XIII.

A *prism* is a solid figure contained by plane figures of which two that are opposite are equal, similar, and parallel to one another; and the others parallelograms.

XIV.

A *sphere* is a solid figure described by the revolution of a semicircle about its diameter, which remains unmoved.

XV.

The *axis* of a sphere is the fixed straight line about which the semicircle revolves.

XVI.

The *centre* of a sphere is the same with that of the semicircle.

XVII.

The *diameter* of a sphere is any straight line which passes through the centre, and is terminated both ways by the superficies of the sphere.

XVIII.

A *cone* is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed.

If the fixed side be equal to the other side containing the right angle, the cone is called a *right-angled cone;* if it be less than the other side, an *obtuse-angled;* and if greater, an *acute-angled cone.*

XIX.

The *axis* of a cone is the fixed straight line about which the triangle revolves.

XX.

The *base* of a cone is the circle described by that side containing the right angle, which revolves.

XXI.

A *cylinder* is a solid figure described by the revolution of a right-angled parallelogram about one of its sides which remains fixed.

XXII.

The *axis* of a cylinder is the fixed straight line about which the parallelogram revolves.

XXIII.

The *bases* of a cylinder are the circles described by the two revolving opposite sides of the parallelogram.

XXIV.

Similar cones and cylinders are those which have their axes and the diameters of their bases proportionals.

XXV.

A *cube* is a solid figure contained by six equal squares.

XXVI.

A *tetrahedron* is a solid figure contained by four equal and equilateral triangles.

XXVII.

An octahedron is a solid figure contained by eight *equal and equilateral* triangles.

XXVIII.

A *dodecahedron* is a solid figure contained by twelve equal pentagons which are equilateral and equiangular.

XXIX.

An *icosahedron* is a solid figure contained by twenty equal and equilateral triangles.

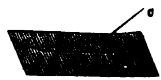
Def. A.

A *parallelopiped* is a solid figure contained by six quadrilateral figures, whereof every opposite two are parallel.

PROP I. THEOR.

One part of a straight line cannot be in a plane, and another part above it.

If it be possible, let AB, part of the straight line ABC, be in the plane, and the part BC above it: and since the straight line AB is in the plane, it can be produced in that plane. Let it be produced to D: and let any plane pass through the



straight line AD, and be turned about it until it pass through the point C; and because the points B, C are in this plane, the straight line* BC is in it: therefore there are two straight lines ABC, ABD in the same plane that have a common segment AB, which is* impossible. Therefore, one part, &c. Q. E. D.

* 7 Def. 1.

*Cor. 11. 1.

PROP. II. THEOR.

Two straight lines which cut one another are in one plane, and three straight lines which meet one another are in one plane.

Let two straight lines AB, CD, cut one another in E; AB, CD shall be one plane: and three straight lines EC, CB, BE which meet one another, shall be in one plane.

Let any plane pass through the straight line EB, and let the plane be turned about EB, produced, if necessary, until it pass through the point C: then because the points E, C are in this plane, the straight line* EC is in it: for the same reason, the straight line BC is in the same; and, by the hypothesis, EB is in it: therefore the three straight lines EC, CB, BE are in one plane: but in the plane in which EC, EB are, in the same are* CD, AB; therefore AB, CD are in one plane. Wherefore two straight lines, &c. Q. E. D.

* 7 Def. 1.

* 1. 11.

PROP. III. THEOR.

If two planes cut one another, their common section is a straight line.

Let two planes AB, BC, cut one another, and let the line DB be their common section: DB shall be a straight line.

If it be not, from the point D to B draw, in the plane AB, the straight line DEB, and in the plane BC the straight line DFB: then two straight lines DEB, DFB have the same extremities, *and therefore* include a

* 10 Ax. 1. space betwixt them; which is* impossible: therefore BD the common section of the planes AB, BC cannot but be a straight line. Wherefore, if two planes, &c. Q. E. D.

PROP. IV. THEOR.

If a straight line stand at right angles to each of two straight lines in the point of their intersection, it shall also be at right angles to the plane which passes through them, that is, to the plane in which they are.

Let the straight line EF stand at right angles to each of the straight lines AB, CD in E, the point of their intersection: EF shall be at right angles to the plane passing through AB, CD.

Take the straight lines AE, EB, CE, ED all equal to one another; and through E draw, in the plane in which are AB, CD, any straight line GEH; and join AD, CB; then, from any point F in EF, draw FA, FG, FD, FC, FH, FB. And because the two straight lines AE, ED are equal to the two BE, EC, and that
* 15. 1. they contain equal angles* AED, BEC, the base AD
* 4. 1. is equal* to the base BC, and the angle DAE to the
* 15. 1. angle EBC: and the angle AEG is equal* to the angle BEH; therefore the triangles AEG, BEH have two angles of the one equal to two angles of the other, each to each, and the sides AE, EB, adjacent to the equal angles, equal to one another: wherefore their other
* 26. 1. sides are equal:* therefore GE is equal to EH, and AG to BH. And because AE is equal to EB, and FE common and at right angles to them, the base AF is
* 4. 1. equal* to the base FB; for the same reason, CF is equal to FD: and because AD is equal to BC, and AF to FB, the two sides FA, AD are equal to the two FB, BC, each to each; and the base DF was proved equal to the base FC; therefore the angle FAD

equal* to the angle FBC. Again, it was proved that * 8. 1.
GA is equal to BH, and also AF to FB; therefore FA
and AG, are equal to FB and BH, each to each; and the angle FAG has been proved equal to the angle FBH; therefore the base GF is equal* to the base FH: again, because it was proved, that GE is equal to EH, and EF is common; therefore GE, EF are equal to HE, EF; and the

* 4. 1.

base GF is equal to the base FH; therefore the angle GEF is equal* to the angle HEF; and consequently * 8. 1.
each of these angles is a right* angle: therefore FE * 10 Def. 1
makes right angles with GH, that is, with any straight line drawn through E in the plane passing through AB, CD. In like manner, it may be proved, that FE makes right angles with every straight line which meets it in that plane: but a straight line is at right angles to a plane when it makes right angles with every straight line which meets it in that plane;* * 3 Def. 11.
therefore EF is at right angles to the plane in which are AB, CD. Wherefore, if a straight line, &c. Q. E. D.

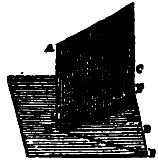
PROP. V. THEOR.

If three straight lines meet all in one point, and a straight line stand at right angles to each of them in that point; these three straight lines are in one and the same plane.

Let the straight line AB stand at right angles to each of the straight lines BC, BD, BE, in B the point where they meet; BC, BD, BE shall be in one and the same plane.

If not, let, if it be possible, BD and BE be in one plane, and BC be above it; and let a plane pass through AB, BC the common section of which with

* 3. 11. the plane in which BD and BE are, is a straight* line; let this be BF: therefore the three straight lines AB, BC, BF are all in one plane, viz. that which passes through AB, BC; and because AB stands at right angles to each of the straight lines BD, BE, it is
* 4. 11. also at right angles* to the plane passing through
* 3 Def. 11. them; and therefore makes right angles* with every straight line meeting it in that plane; but BF which is in that plane meets it: therefore the angle ABF is a right angle; but the angle ABC, by the hypothesis, is also a right angle; therefore the angle ABF is equal to the angle



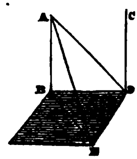
* 9 Ax. ABC, and they are both in the same plane, which is* impossible: therefore the straight line BC is not above the plane in which are BD and BE: wherefore the three straight lines BC, BD, BE are in one and the same plane. Therefore, if three straight lines, &c. Q. E. D.

PROP. VI. THEOR.

If two straight lines be at right angles to the same plane, they shall be parallel to one another.

Let the straight lines AB, CD be at right angles to the same plane; AB shall be parallel to CD.

Let them meet the plane in the points B, D, and draw the straight
* 11. 1. line BD, to which draw* DE at right angles, in the same plane;
* 1. 3. and make* DE equal to AB, and join BE, AE, AD. Then, because AB is perpendicular to the plane,



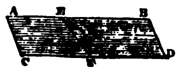
* 3 Def. 11. it makes right* angles with every straight line which meets it, and is in that plane: but BD, BE, which are in that plane, do each of them meet AB: therefore

each of the angles ABD, ABE is a right angle. For the same reason, each of the angles CDB, CDE is a right angle: and because AB is equal to DE, and BD common, the two sides AB, BD are equal to the two ED, DB, each to each; and they contain right angles; therefore the base AD is equal* to the base BE. * 4. 1.
Again, because AB is equal to DE, and BE to AD; AB, BE are equal to ED, DA, each to each; and, in the triangles ABE, EDA, the base AE is common; therefore the angle ABE is equal* to the angle EDA: * 8. 1.
but ABE is a right angle; therefore EDA is also a right angle, and ED perpendicular to DA; but it is also perpendicular to each of the two BD, DC: wherefore ED is at right angles to each of the three straight lines BD, DA, DC in the point in which they meet; therefore* these three straight lines are all in the same * 5. 11.
plane: but AB is in the plane in which are BD, DA, because* any three straight lines which meet one ano- * 2. 11.
ther are in one plane; therefore AB, BD, DC are in one plane: and each of the angles ABD, BDC is a right angle; therefore AB is parallel* to CD Where- * 28. 1.
fore, if two straight lines, &c. Q. E. D.

PROP. VII. THEOR.

If two straight lines be parallel, the straight line drawn from any point in the one to any point in the other, is in the same plane with the parallels.

Let AB, CD be parallel straight lines, and take any point E in the one, and the point F in the other: the straight line which joins E and F shall be in the same plane with the parallels.



If not, let it be, if possible, above the plane, as EGF; and in the plane ABCD in which the parallels are, draw the straight

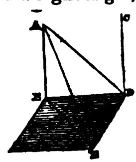
line EHF from E to F; and since EGF also is a straight line, the two straight lines EHF, EGF include
* 10 Ax. 1. a space between them, which is* impossible: therefore the straight line joining the points E, F is not above the plane in which the parallels AB, CD are, and is therefore in that plane. Wherefore, if two straight lines, &c. Q. E. D.

PROP. VIII. THEOR.

If two straight lines be parallel, and one of them is at right angles to a plane; the other also shall be at right angles to the same plane.

Let AB, CD be two parallel straight lines, and let one of them AB be at right angles to a plane; the other CD shall be at right angles to the same plane.

Let AB, CD meet the plane in the points B, D, and
* 7. 11. join BD: therefore* AB, CD, BD are in one plane.
* 11. 1. In the plane to which AB is at right angles, draw* DE
* 3. 1. at right angles to BD, and make* DE equal to AB, and join BE, AE, AD. And because AB is perpendicular to the plane, it is perpendicular to every straight line
* 3 Def. 11. which meets it, and is in that plane:* therefore each of the angles ABD, ABE, is a right angle: and because the straight line BD meets the parallel straight lines AB, CD, the angles ABD, CDB are together
* 29. 1. equal* to two right angles: and ABD is a right angle; therefore also CDB is a right angle, and CD perpendicular to BD: and because AB is equal to DE, and BD common, the two AB, BD are equal to the two ED, DB, each to each; and the angle ABD is equal to the angle EDB, because each of them is



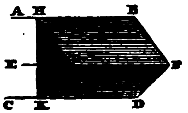
* 4. 1. a right angle; therefore the base AD is equal* to the base BE. Again, because AB is equal to DE, and

BE to AD; the two AB, BE are equal to the two ED, DA, each to each; and the base AE is common to the triangles ABE, EDA; therefore the angle ABE is equal* to the angle EDA: and ABE is a right angle; * 8. 1.
therefore EDA is a right angle, and ED perpendicular to DA: but it is also perpendicular* to BD; therefore * Const.
ED is perpendicular* to the plane which passes * 4. 11.
through BD, DA, and therefore* makes right angles * 3 Def. 11.
with every straight line meeting it in that plane: but DC is in the plane passing through BD, DA, because all three are in the plane in which are the parallels AB, CD: wherefore ED is at right angles to DC; and therefore CD is at right angles to DE: but CD is also at right angles to DB; therefore CD is at right angles to the two straight lines DE, DB in D the point of their intersection; and therefore is at right angles* to * 4. 11.
the plane passing through DE, DB, which is the same plane to which AB is at right angles. Therefore, if two straight lines, &c. Q. E. D.

PROP. IX. THEOR.

Two straight lines which are each of them parallel to the same straight line, and not in the same plane with it, are parallel to one another.

Let AB, CD be each of them parallel to EF, and not in the same plane with it: AB shall be parallel to CD.



In EF take any point G, from which* draw, in the plane passing through EF, AB, the straight line GH at right angles to EF; * 11. 1.
and in the plane passing through EF, CD, draw GK at right angles to the same EF. *And because EF is* perpendicular both to GH and GK, *EF is* perpendicular* to the plane HGK passing * 4. 11

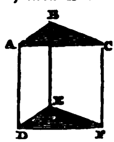
through them: and EF is parallel to AB; therefore
* 8. 11. AB is at right angles* to the plane HGK: for the same reason, CD is likewise at right angles to the plane HGK; therefore AB, CD are each of them at right angles to the plane HGK. But if two straight lines are at right angles to the same plane, they are
* 6. 11. parallel* to one another: therefore AB is parallel to CD. Wherefore, two straight lines, &c. Q. E. D.

PROP. X. THEOR.

If two straight lines meeting one another be parallel to two others that meet one another, and are not in the same plane with the first two; the first two and the other two shall contain equal angles.

Let the two straight lines AB, BC, which meet one another, be parallel to the two straight lines DE, EF, that meet one another, and are not in the same plane with AB, BC: the angle ABC shall be equal to the angle DEF.

Take BA, BC, ED, EF all equal to one another; and join AD, CF, BE, AC, DF: then, because BA is
* 33. 1. equal and parallel to ED, therefore AD is* both equal and parallel to BE: for the same reason, CF is equal and parallel to BE; therefore AD and CF are each of them equal and parallel to BE. But straight lines that are parallel to the same straight line, and not in
* 9. 11. the same plane with it, are parallel* to one another: therefore AD is parallel
* 1 Ax. 1 to CF; and it is equal* to it; and AC, DF join them towards the same parts:
* 33. 1. and therefore* AC is equal and parallel to DF. And because AB, BC are equal to DE, EF, each to each, and the base AC to



* 8. 1. the base DF; the angle ABC is equal* to the angle DEF. Therefore, if two straight lines, &c. Q. E. D.

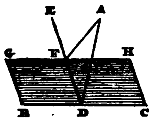
PROP. XI. PROB.

To draw a straight line perpendicular to a plane, from a given point above it.

Let A be the given point above the plane BH; it is required to draw from the point A a straight line perpendicular to the plane BH.

In the plane draw any straight line BC, and from the point A draw* AD perpendicular to BC: if then * 12. 1.
AD be also perpendicular to the plane BH, the thing required is already done; but if it be not, from the point D draw,* in the plane BH, the straight line DE * 11. 1.
at right angles to BC; and from the point A draw AF perpendicular to DE: AF shall be perpendicular to the plane BH.

Through F draw* GH parallel to BC: and because * 31. 1.
BC is at right angles to ED and DA, BC is at right angles* to the plane passing through ED, DA: and * 4. 11.
GH is parallel to BC: but, if two straight lines be parallel, one of which is at right angles to a plane, the other is at right* angles to the same plane; * 8. 11.
wherefore GH is at right angles to the plane through ED, DA; and is perpendicular* to every straight * 3 Def. 11.
line meeting it in that plane: but AF, which is in the plane through ED, DA, meets it; therefore GH is perpendicular to AF; and consequently AF is perpendicular to GH; and AF is perpendicular to DE; therefore AF is perpendicular to each of the straight lines GH, DE. But if a straight line stand at right angles to each of two straight lines in the point of their intersection, it is also at right angles* to the plane * 4. 11.
passing through them: but the plane passing through ED, GH is the plane BH; therefore AF is perpen-

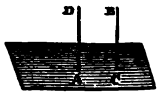


dicular to the plane BH; therefore, from the given point A, above the plane BH, the straight line AF is drawn perpendicular to that plane. Which was to be done.

PROP. XII. PROB.

To erect a straight line at right angles to a given plane, from a point given in the plane.

Let A be the point given in the plane; it is required to erect a straight line from the point A at right angles to the plane.



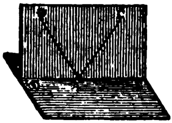
From any point B above the
* 11. 11. plane draw* BC perpendicular to
* 31. 1. it; and from A draw* AD parallel to BC. Because, therefore, AD, CB are two parallel straight lines, and one of them BC is at right angles to the
* 8. 11. given plane, the other AD is also* at right angles to it: therefore a straight line has been erected at right angles to a given plane from a point given in it. Which was to be done.

PROP. XIII. THEOR.

From the same point in a given plane, there cannot be two straight lines at right angles to the plane, upon the same side of it: and there can be but one perpendicular to a plane from a point above the plane.

For, if it be possible, let the two straight lines AB, AC, be at right angles to a given plane from the same point A in the plane, and upon the same side of it; and let a plane pass through BA, AC; the common section of this with the given plane is a
* 3. 11. straight* line passing through A: let DAE be their common section: therefore the straight lines AB, AC,

DAE are in one plane: and because CA is at right angles to the given plane, it makes right angles* with every straight line meeting it in that plane: but DAE, which is in that plane, meets CA; therefore CAE is a right angle. For the same reason BAE is a right angle: wherefore the angle CAE is equal* to the angle BAE; but this is impossible, because they are in one plane. Also, from a point above a plane, there can be but one perpendicular to that plane; for, if there could be two, they would be parallel* to one another, which is absurd. Therefore, from the same point, &c. Q. E. D.



* 3 Def. 11.

* 11 Ax.

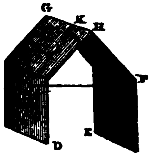
* 6. 11.

PROP. XIV. THEOR.

Planes to which the same straight line is perpendicular, are parallel to one another.

Let the straight line AB be perpendicular to each of the planes CD, EF; these planes shall be parallel to one another.

If not, they will meet one another when produced; let them meet; then their common section is a straight line GH, in which take any point K, and join AK, BK. Then, because AB is perpendicular to the plane EF, it is perpendicular* to the straight line BK which is in that plane: therefore ABK is a right angle. For the same reason, BAK is a right angle; wherefore the two angles ABK, BAK of the triangle ABK are equal to two right angles, which is* impossible: therefore the planes CD, EF, though produced, *do not* meet one another; that is, *they are parallel.** Therefore planes, &c. Q. E. D.



* 3 Def. 11.

* 17. 1.

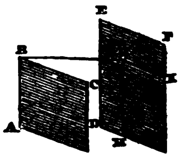
* 8 Def. 11.

PROP. XV. THEOR.

If two straight lines meeting one another, be parallel to two other straight lines which meet one another, but are not in the same plane with the first two: the plane which passes through these is parallel to the plane passing through the others.

Let AB, BC, two straight lines meeting one another, be parallel to DE, EF two other straight lines that meet one another, but are not in the same plane with AB, BC: the planes through AB, BC, and DE, EF shall not meet, though produced.

* 11. 11. From the point B draw BG perpendicular* to the plane which passes through DE, EF, and let it meet
* 31. 1. that plane in G; and through G draw GH parallel* to ED, and GK parallel to EF. And because BG is
* 3 Def. 11. perpendicular to the plane through DE, EF, it makes* right angles with every straight line meeting it in that plane: but the straight lines GH, GK in that plane meet it; therefore each of the angles BGH, BGK is a right angle: and because



* 9. 11. BA is parallel* to GH (for each of them is parallel to DE, and they are not both in the same plane with it) the angles GBA, BGH are together
* 29. 1. equal* to two right angles: and BGH is a right angle; therefore also GBA is a right angle, and GB perpendicular to BA. For the same reason, GB is perpendicular to BC: therefore since the straight line GB stands at right angles to the two straight lines BA,
* 4. 11. BC, which cut one another in B; GB is perpendicular*
* Const. to the plane through BA, BC: and it is perpendicular* to the plane through DE, EF; therefore BG is perpendicular to each of the planes through AB, BC, and

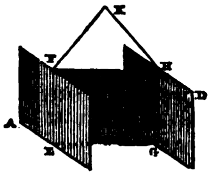
EF: but planes to which the same straight line
rpendicular, are parallel* to one another: there- *14. 11.
the plane through AB, BC is parallel to the plane
igh DE, EF. Wherefore, if two straight lines,
Q. E. D.

PROP. XVI. THEOR.

o parallel planes be cut by another plane, their common sections with it are parallels.

t the parallel planes AB, CD be cut by the plane
G, and let their common sections with it be EF,
EF shall be parallel to GH.

r, if it is not, EF, GH shall meet, if produced,
: on the side of FH, or EG. First, let them be
iced on the side of FH, and meet in the point K:
fore, since EFK is in the plane AB, every point* *1. 11.
FK is in that plane;
K is a point in EFK;
ore K is in the plane
For the same reason
ilso in the plane CD:
fore the planes AB,
roduced meet one ano-
but they do not meet,



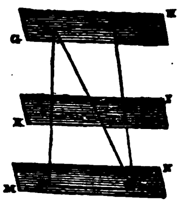
they are parallel by the hypothesis: therefore
raight lines EF, GH do not meet when produced
: side of FH. In the same manner it may be
l, that EF, GH do not meet when produced on
de of EG: but straight lines which are in the
plane and do not meet, though produced either
are parallel: therefore EF is parallel to GH.
fore, if two parallel planes, &c. Q. E. D.

PROP. XVII. THEOR.

If two straight lines be cut by parallel planes, they shall be cut in the same ratio.

Let the straight lines AB, CD be cut by the parallel planes GH, KL, MN, in the points A, E, B; C, F, D; as AE is to EB, so shall CF be to FD.

Join AC, BD, AD, and let AD meet the plane KL in the point X; and join EX, XF. Because the two parallel planes KL, MN are cut by the plane EBDX,
* 16. 11. the common sections EX, BD, are* parallel: for the same reason, because the two parallel planes GH, KL are cut by the plane AXFC, the common sections AC, XF are parallel: and because EX is parallel to BD, a side of the triangle ABD,
* 2. 6. as AE to EB, so is* AX to XD. Again, because XF is parallel to AC, a side of the triangle ADC, as AX to XD, so is CF to FD: and it was
* 11. 5. proved that AX is to XD, as AE to EB; therefore,* as AE to EB, so is CF to FD. Wherefore, if two straight lines, &c. Q. E. D.



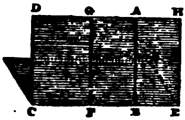
PROP. XVIII. THEOR.

If a straight line be at right angles to a plane, every plane which passes through it shall be at right angles to that plane.

Let the straight line AB be at right angles to the plane CK; every plane which passes through AB shall be at right angles to the plane CK.

Let any plane DE pass through AB, and let CE be the common section of the planes DE, CK; take

any point F in CE, from which draw FG in the plane DE at right* angles to CE; and because AB is perpendicular to the plane CK, therefore* it is also perpendicular to every straight line in that plane meeting it: and consequently it is perpendicular to CE: wherefore ABF is a right angle; but GFB is likewise a right angle: therefore AB is parallel* to FG: and AB is at right angles to the plane CK; therefore FG is also* at right angles to the same plane. But one plane is at right angles to another plane when the straight lines drawn in one of the planes, at right angles to their common section, are also* at right angles to the other plane; and any straight line FG in the plane DE, which is at right angles to CE the common section of the planes, has been proved to be perpendicular to the other plane CK; therefore the plane DE is at right angles to the plane CK. In like manner, it may be proved that all the planes which pass through AB are at right angles to the plane CK. Therefore, if a straight line, &c. Q. E. D.



* 11. 1.
* 3 Def. 11.
* 28. 1.
* 8. 11.
* 4 Def. 11.

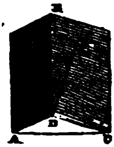
PROP. XIX. THEOR.

If two planes which cut one another be each of them perpendicular to a third plane; their common section is perpendicular to the same plane.

Let the two planes AB, BC be each of them perpendicular to a third plane, and let BD be the common section of the first two; BD shall be perpendicular to the third plane.

If it be not, from the point D, in the plane AB, draw * the straight line DE at right angles to AD the * 11. 1.
common section of the plane AB with the third plane; *and in the plane BC* draw DF at right angles to CD

the common section of the plane BC with the third plane. And because the plane AB is perpendicular to the third plane, and DE is drawn in the plane AB at right angles to AD their common section,



* 4 Def. 11. DE is perpendicular* to the third plane. In the same manner it may be proved that DF is perpendicular to the third plane: wherefore, from the point D two straight lines stand at right angles to the third plane, upon the same side

* 13. 11. of it, which is* impossible; therefore, from the point D there cannot be any straight line at right angles to the third plane, except BD the common section of the planes AB, BC: therefore BD is perpendicular to the third plane. Wherefore, if two planes, &c. Q. E. D.

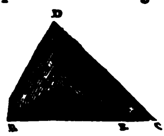
PROP. XX. THEOR.

If a solid angle be contained by three plane angles, any two of them are greater than the third.

Let the solid angle at A be contained by the three plane angles BAC, CAD, DAB: any two of them shall be greater than the third.

If the angles BAC, CAD, DAB be all equal, it is evident that any two of them are greater than the third. But if they are not, let BAC be that angle which is not less than either of the other two, and is greater than one of them DAB; and at the point A in the straight line AB, make, in the plane which passes

* 23. 1. through BA, AC, the angle BAE equal* to the angle DAB; and make AE equal to AD, and through E draw BEC cutting AB, AC in the points B, C, and join DB, DC. And because DA is equal to AE, and AB is common, the two DA,



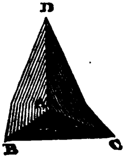
AB are equal to the two EA, AB, each to each, and the angle DAB is equal to the angle EAB: therefore the base DB is equal* to the base BE. And because * 4. 1.
BD, DC are greater* than CB, and one of them BD * 20. 1.
has been proved equal to BE, a part of CB, therefore the other DC is greater* than the remaining part EC. * 5 Ax.
And because DA is equal to AE, and AC common, but the base DC greater than the base EC; therefore the angle DAC is greater* than the angle EAC; and, by * 25. 1.
the construction, the angle DAB is equal to the angle BAE; wherefore the angles DAB, DAC are together greater* than BAE, EAC, that is, than the angle BAC: * 4 Ax.
but BAC is not less than either of the angles DAB, DAC; therefore BAC, with either of them, is greater than the other. Wherefore, if a solid angle, &c. Q. E. D.

PROP. XXI. THEOR.

Every solid angle is contained by plane angles which together are less than four right angles.

First, let the solid angle at A be contained by three plane angles BAC, CAD, DAB: these three together shall be less than four right angles.

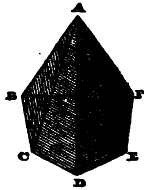
Take in each of the straight lines AB, AC, AD any points B, C, D, and join BC, CD, DB. Then, because the solid angle at B is contained by the three plane angles CBA, ABD, DBC, any two of them are greater* * 20. 11.
than the third; therefore the angles CBA, ABD are greater than the angle DBC: for the same reason, the angles BCA, ACD are greater than the angle DCB; and the angles CDA, ADB greater than BDC: wherefore the six angles CBA, ABD, BCA, ACD, CDA, ADB are greater than the three angles DBC, BCD, CDB: but the three angles DBC, BCD, CDB are equal* to two right * 32. 1.



angles: therefore the six angles CBA, ABD, BCA, ACD, CDA, ADB are greater than two right angles. And because the three angles of each of the triangles ABC, ACD, ADB are equal to two right angles, therefore the nine angles of these three triangles, viz. the angles CBA, BAC, ACB, ACD, CDA, DAC, ADB, DBA, BAD are equal to six right angles: of these the six angles CBA, ACB, ACD, CDA, ADB, DBA are greater than two right angles: therefore the remaining three angles BAC, DAC, BAD, which contain the solid angle at A, are less than four right angles.

Next, let the solid angle at A be contained by any number of plane angles BAC, CAD, DAE, EAF, FAB; these together shall be less than four right angles.

Let the planes in which the angles are, be cut by a plane, and let the common section of it with those planes be BC, CD, DE, EF, FB. And because the solid angle at B is contained by three plane angles CBA, ABF, FBC, of which any two are greater* than the third, the angles CBA, ABF are greater than the angle FBC: for the same reason, the two plane angles at each of the points C, D, E, F, viz. the angles which are at the bases of the triangles having the common vertex A, are greater than the third angle at the same point, which is one of the angles of the polygon BCDEF: therefore all the angles at the bases of the triangles are together greater than all the angles of the polygon: and because all the angles of the triangles are together equal to twice as many right angles* as there are triangles; that is, as there are sides in the polygon BCDEF; and that all the angles of the polygon, together with four right angles, are likewise



* 20. 11.

* 32. 1.

equal to twice as many right angles* as there are sides in the polygon; therefore all the angles of the triangles are equal* to all the angles of the polygon together with four right angles. But all the angles at the bases of the triangles are greater than all the angles of the polygon, as has been proved: wherefore the remaining angles of the triangles, viz. those at the vertex, which contain the solid angle at A, are less than four right angles. Therefore, every solid angle, &c. Q. E. D. * 1 Cor. 32. 1. * 1 Ax

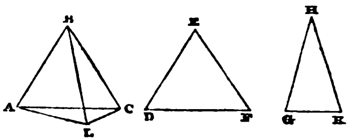
PROP. XXII. THEOR.

If every two of three plane angles be greater than the third, and if the straight lines which contain them be all equal; a triangle may be made of the straight lines that join the extremities of those equal straight lines.

Let ABC, DEF, GHK be three plane angles, whereof every two are greater than the third, and are contained by the equal straight lines AB, BC, DE, EF, GH, HK; if their extremities be joined by the straight lines AC, DF, GK, a triangle may be made of three straight lines equal to AC, DF, GK; that is, every two of them shall be together greater than the third.

If the angles at B, E, H are equal: AC, DF, GK are also equal,* and any two of them greater than the third: but if the angles are not all equal, let the angle ABC be not less than either of the two at E, H; therefore the straight line AC is not less* than either of the other two DF, GK; and it is plain that AC, together with either of the other two, must be greater than the third. Also DF with GK are greater than AC: for, at the point B in the straight line AB make* the angle ABL equal to the angle GHK, and make BL equal to one of the straight lines AB, BC, DE, EF, GH, HK, * 4. 1. * 4. 1. or 24. 1. * 23. 1.

and join AL, LC: then because AB, BL are equal to GH, HK, and the angle ABL to the angle GHK, the
* 4. 1. base AL is equal* to the base GK: and because the
* Hyp. angles at E, H are greater* than the angle ABC, of which the angle at H is equal to ABL, therefore the
* 5 Ax. remaining angle at E is greater* than the angle LBC:



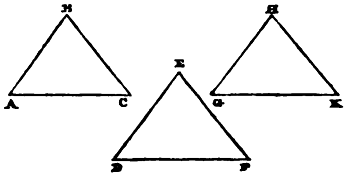
and because the two sides LB, BC are equal to the two DE, EF, and that the angle DEF is greater than
* 24. 1. the angle LBC, the base DF is greater* than the base LC: and it has been proved that GK is equal to AL;
* 4 Ax. therefore DF and GK are greater* than AL and LC:
* 20. 1. but AL and LC are greater* than AC; much more then are DF and GK greater than AC. Wherefore every two of these straight lines AC, DF, GK are greater than the third; and, therefore, a triangle may
* 22. 1. be made,* the sides of which shall be equal to AC, DF, GK. Q. E. D.

PROP. XXIII. PROB.

To make a solid angle which shall be contained by three
* 20. 11. *given plane angles, any two of them being greater**
* 21. 11. *than the third, and all three together less* than four right angles.*

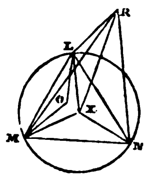
Let the three given plane angles be ABC, DEF, GHK, any two of which are greater than the third, and all of them together less than four right angles: it is required to make a solid angle contained by three plane angles equal to ABC, DEF, GHK, each to each.

From the straight lines containing the angles, cut off AB, BC, DE, EF, GH, HK, all equal to one another; and join AC, DF, GK: then a triangle may be made* of three straight lines equal to AC, DF, GK. * 22. 11.



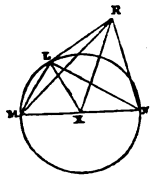
Let this be the triangle LMN,* so that AC be equal to LM, DF to MN, and GK to LN; and about the triangle LMN describe* a circle, and find X its centre, which will be either within or out of the triangle, or in one of its sides. * 22. 1. * 5. 4.

First, let X the centre be within the triangle, and join LX, MX, NX: AB shall be greater than LX. If not, AB must either be equal to LX, or less than it. First, let it be equal: then because AB is equal to LX, and that AB is also equal to BC, and LX to XM, AB and BC are equal to LX and XM, each to each; and the base AC is, by construction, equal to the base LM; wherefore the angle ABC is equal* to the angle LXM. For the same reason, the angle DEF is equal to the angle MXN, and the angle GHK to the angle NXL: therefore the three angles ABC, DEF, GHK are equal to the three angles LXM, MXN, NXL: but the three angles LXM, MXN, NXL are equal* to four right angles: therefore also the three angles ABC, DEF, GHK are *equal to four right* angles: but this is absurd, because, *by the hypothesis,* they are less than four right * 8. 1. * 2. Cor. 15. 1.



angles; therefore AB is not equal to LX. But neither can AB be less than LX: for, if possible, let it be less, and upon the straight line LM, on the side of it on
* 22. 1. which is the centre X, describe* the triangle LOM, the sides LO, OM of which are equal to AB, BC; and because the base LM is equal to the base AC, the angle
* 8. 1. LOM is equal* to the angle ABC: and AB, that is, LO, by the hypothesis, is less than LX; wherefore LO, OM fall within the triangle LXM; for, if they
* 21. 1. fell upon its sides, or without it, they would be* equal to, or greater than LX, XM: therefore the angle
* 21. 1. LOM, that is, the angle ABC, is greater* than the angle LXM. In the same manner it may be proved that the angle DEF is greater than the angle MXN, and the angle GHK greater than the angle NXL: therefore the three angles ABC, DEF, GHK are greater
* 2 Cor. 15. 1. than the three angles LXM, MXN, NXL; that is,* than four right angles: but this is absurd, because the
* Hyp. same angles ABC, DEF, GHK are less* than four right angles: therefore AB is not less than LX, and it has been proved that it is not equal to LX; therefore AB is greater than LX.

Next, let X the centre of the circle fall in one of the sides of the triangle, viz. in MN, and join XL. In this case also AB shall be greater than LX. If not, AB is either equal to LX, or less than it. First, let it be equal to XL: therefore AB and BC, that is, DE, and EF, are equal to MX and XL, that is, to MN: but, by the construction, MN is equal to DF; therefore DE, EF

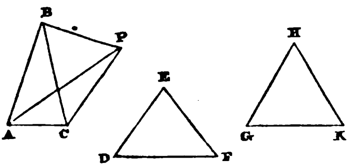


* 20. 1. are equal to DF, which is* impossible: wherefore AB is not equal to LX; nor is it less; for then, much more, an absurdity would follow: therefore AB is greater than LX.

But, let X the centre of the circle fall without the triangle LMN, and join LX, MX, NX. In this case likewise AB shall be greater than LX. If not, it is either equal to, or less than LX. First, let it be equal; it may be proved in the same manner, as in the first case, that the angle ABC is equal to the angle MXL, and GHK to LXN; therefore the whole angle MXN is equal to the two angles ABC, GHK: but ABC and GHK are together greater* than the angle DEF; therefore also the angle MXN is greater than DEF. And because DE, EF are equal to MX, XN, and the base DF to the base MN, the angle MXN is equal* to the angle DEF; but this is absurd, because it has been proved that MXN is greater than DEF: therefore AB is not equal to LX. Nor yet is it less; for then, as has been proved in the first case, the angle ABC is greater than the angle MXL, and the angle GHK greater than the angle LXN. At the point B in the straight line CB make the angle CBP equal to the angle GHK, and make BP equal to HK, and join CP, AP. And because CB is equal to GH; CB, BP are

* Hyp.

* 8. 1.



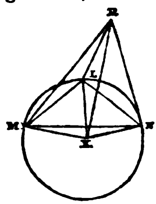
equal to GH, HK, each to each, and they contain equal angles; wherefore the base CP is equal to the base GK, that is, to LN. And in the isosceles triangles ABC, MXL, because the angle ABC is greater than the angle MXL, therefore the angle MLX at the base is *greater* * *than the angle* ACB at the base. For the *same reason,* because the angle GHK, or CBP, is

* 32. 1.

greater than the angle LXN, the angle XLN is greater than the angle BCP: therefore the whole angle MLN is greater than the whole angle ACP. And because ML, LN are equal to AC, CP, each to each, but the angle MLN is greater than the angle ACP, the base

* 24. 1. MN is greater* than the base AP: and MN is equal to DF; therefore also DF is greater than AP. Again, because DE, EF are equal to AB, BP, each to each, but the base DF greater than the base AP, the an-

* 25. 1. gle DEF is greater* than the angle ABP: but ABP is equal to the two angles ABC, CBP, that is, to the two angles ABC, GHK; therefore the angle DEF is greater than the two angles ABC, GHK; but this is impossible, because, by hypothesis, it is less than the same two angles; therefore AB is not less than LX; and it has been proved that it is not equal to it: therefore AB is greater than LX.



* 12. 11. From the point X erect* XR at right angles to the plane of the circle LMN. And because it has been proved in all the cases, that AB is greater than LX, find a square equal to the excess of the square of AB above the square of LX, and make RX equal to its side, and join RL, RM, RN; then the solid angle at R shall be the angle required.

Because RX is perpendicular to the plane of the

* 3 Def. 11. circle LMN, it is* perpendicular to each of the straight lines LX, MX, NX. And because LX is equal to MX, and XR common, and at right angles to each of

* 4. 1. them, the base RL is equal* to the base RM. For the same reason, RN is equal to each of the two RL, RM: therefore the three straight lines RL, RM, RN are all equal. And because the square of XR is equal to the excess of the square of AB above the square of LX;

therefore the square of AB is equal to the squares of LX, XR: but the square of RL is equal* to the same squares, because LXR is a right angle: therefore the square of AB is equal to the square of RL, and the straight line AB to RL. But each of the straight lines BC, DE, EF, GH, HK is equal to AB, and each of the two RM, RN is equal to RL: wherefore AB, BC, DE, EF, GH, HK are each of them equal to each of the straight lines RL, RM, RN. And because RL, RM, are equal to AB, BC, each to each, and the base LM to the base AC; the angle LRM is equal* to the angle ABC. For the same reason, the angle MRN is equal to the angle DEF, and NRL to GHK. Therefore there is made a solid angle at R, which is contained by three plane angles LRM, MRN, NRL, which are equal to the three given plane angles ABC, DEF, GHK, each to each. Which was to be done.

* 47. 1.

* 8. 1.

PROP. A. THEOR.

If each of two solid angles be contained by three plane angles equal to one another, each to each; the planes in which the equal angles are, have the same inclination to one another.

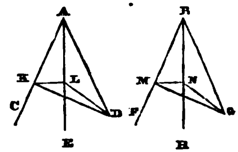
Let there be two solid angles at the points A, B; and let the angle at A be contained by the three plane angles CAD, CAE, EAD; and the angle at B by the three plane angles FBG, FBH, HBG; of which the angle CAD is equal to the angle FBG, and CAE to FBH, and EAD to HBG: the planes in which the equal angles are, shall have the same inclination to one another.

In the straight line AC take any point K, and in the plane CAD draw* from K the straight line KD at right angles to AC, and in the plane CAE the straight line KL at right angles to the same AC: therefore the angle DKL is the inclination* of the plane CAD to

* 11. 1.

* 6 Def. 11.

the plane CAE. In BF take BM equal to AK, and in the planes FBG, FBH, draw from the point M the straight lines MG, MN at right angles to BF; there-
6 Def. fore the angle GMN is the inclination* of the plane FBG to the plane FBH. Join LD, NG; and because in the triangles KAD, MBG, the angles KAD, MBG are equal, as also the right angles AKD, BMG, and that



the sides AK, BM, adjacent to the equal angles, are
* 26. 1. equal to one another: therefore KD is equal* to MG, and AD to BG. For the same reason, in the triangles KAL, MBN, KL is equal to MN, and AL to BN: and in the triangles LAD, NBG, LA, AD are equal to NB, BG, each to each, and they contain equal angles;
* 4. 1. therefore the base LD is equal* to the base NG. Lastly, in the triangles KLD, MNG, the sides DK, KL are equal to GM, MN, each to each, and the base LD to the base NG; therefore the angle DKL is
* 8. 1. equal* to the angle GMN: but the angle DKL is the inclination of the plane CAD to the plane CAE, and the angle GMN is the inclination of the plane FBG to the plane FBH, which planes have therefore the same
* 7 Def. 11. inclination* to one another. And in the same manner it may be demonstrated, that the other planes in which the equal angles are, have the same inclination to one another. Therefore, if each of two solid angles, &c. Q. E. D.

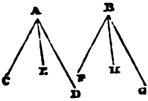
PROP. B. THEOR.

Two solid angles contained, each by three plane angles which are equal to one another, each to each, and alike situated, are equal to one another.

Let there be two solid angles at A and B, of which the solid angle at A is contained by the three plane

angles CAD, CAE, EAD; and that at B by the three plane angles FBG, FBH, HBG; of which CAD is equal to FBG; CAE to FBH; and EAD to HBG: the solid angle at A shall be equal to the solid angle at B.

Let the solid angle at A be applied to the solid angle at B; and, first, the plane angle CAD being applied to the plane angle FBG, so that the point A may coincide with the point B, and the straight line AC with BF; then AD coincides with BG, because the angle CAD is equal to the angle FBG. And because the inclination of the plane CAE to the plane CAD is equal* to the inclination of the plane FBH to the plane FBG, the plane CAE coincides with the plane FBH, because the planes CAD, FBG coincide with one another. And because the straight lines AC, BF coincide, and that the angle CAE is equal to the angle FBH; therefore AE coincides with BH, and AD coincides with BG; wherefore the plane EAD coincides with the plane HBG: therefore the solid angle A coincides with the solid angle B, and consequently they are equal* to one another. Q. E. D.



* A. 11.

* 8 Ax. 1.

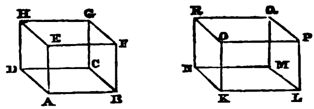
PROP. C. THEOR.

Solid figures contained by the same number of equal and similar planes alike situated, and having none of their solid angles contained by more than three plane angles; are equal and similar to one another.

Let AG, KQ be two solid figures contained by the same number of similar and equal planes, alike situated, viz. let the plane AC be similar and equal to the plane KM, the plane AF to KP; BG to LQ; GD to QN; DE to NO; and lastly, FH similar and equal to PR:

the solid figure AG shall be equal and similar to the solid figure KQ.

Because the solid angle at A is contained by the three plane angles BAD, BAE, EAD, which, by the hypothesis, are equal to the plane angles LKN, LKO, OKN, which contain the solid angle at K, each to
* B. 11. each; therefore the solid angle at A is equal* to the solid angle at K. In the same manner, the other solid angles of the figures are equal to one another. Let, then, the solid figure AG be applied to the solid figure



KQ, and first, the plane figure AC being applied to the plane figure KM, so that the straight line AB may coincide with KL, the figure AC must coincide with the figure KM, because they are equal and similar: therefore the straight lines AD, DC, CB coincide with KN, NM, ML, each with each; and the points A, D, C, B, with the points K, N, M, L: and the solid angle at
* B. 11. A coincides with* the solid angle at K; wherefore the plane AF coincides with the plane KP, and the figure AF with the figure KP, because they are equal and similar to one another: therefore the straight lines AE, EF, FB, coincide with KO, OP, PL; and the points E, F, with the points O, P. In the same manner, the figure AH coincides with the figure KR, and the straight line DH with NR, and the point H with the point R. And because the solid angle at B is equal to the solid angle at L, it may be proved, in the same manner, that the figure BG coincides with the figure LQ, and the straight line CG with MQ, and the point G with the point Q. Therefore, since all the

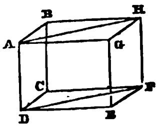
planes and sides of the solid figure AG coincide with the planes and sides of the solid figure KQ, AG is equal and similar to KQ. And, in the same manner, any other solid figures whatever contained by the same number of equal and similar planes, alike situated, and and having none of their solid angles contained by more than three plane angles, may be proved to be equal and similar to one another. Q. E. D.

PROP. XXIV. THEOR.

If a solid be contained by six planes, two and two of which are parallel; the opposite planes are similar and equal parallelograms.

Let the solid CDGH be contained by the parallel planes AC, GF: BG, CE; FB, AE: its opposite planes shall be similar and equal parallelograms.

Because the two parallel planes BG, CE, are cut by the plane AC, their common sections AB, CD,* are parallel: and because the two parallel planes BF, AE, are cut by the plane AC, their common sections AD, BC,* are parallel: and AB is parallel to CD; therefore AC is a parallelogram. In like manner, it may be proved that each of the figures CE, FG, GB, BF, AE is a parallelogram. Join AH, DF; and because AB is parallel to DC, and BH to CF; the two straight lines AB, BH, which meet one another, are parallel to DC and CF which meet one another, and are not in the same plane with the other two; wherefore they contain* equal angles; therefore the angle ABH is equal to the angle DCF. And because AB, BH, are equal to DC, CF, and the angle ABH equal to *the angle DCF:* therefore the base AH is *equal* to the base DF,* and the triangle ABH to the



* 16. 11.
* 16. 11.
* 10. 11.
* 4. 1.

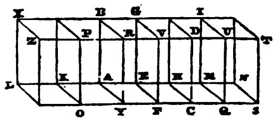
* 34. 1. triangle DCF: but the parallelogram BG is double* of the triangle ABH, and the parallelogram CE double of the triangle DCF; therefore the parallelogram BG is equal and similar to the parallelogram CE. In the same manner it may be proved, that the parallelogram AC is equal and similar to the parallelogram GF, and the parallelogram AE to BF. Therefore, if a solid, &c. Q. E. D.

PROP. XXV. THEOR.

If a parallelopiped be cut by a plane parallel to two of its opposite planes; it divides the whole into two solids, the base of one of which is to the base of the other, as the one solid is to the other.

Let the parallelopiped ABCD be cut by the plane EV, which is parallel to the opposite planes AR, HD, and divide the whole into the two solids ABFV, EGCD; the base AEFY of the first shall be to the base EHCF of the other, as the solid ABFV to the solid EGCD.

Produce AH both ways, and take any number of straight lines HM, MN, each equal to EH, and any number AK, KL each equal to EA, and complete the parallelograms LO, KY, HQ, MS, and the solids LP, KR, HU, MT. Then, because the straight lines LK,



KA, AE are all equal, the parallelograms LO, KY,
* 36. 1. AF are* equal: and likewise the parallelograms
* 24. 11. KX, KB, AG: also* the parallelograms LZ, KP, AR, are equal, because they are opposite planes: for the same reason, the parallelograms EC, HQ, MS, are

equal;* and the parallelograms HG, HI, IN, as also * 36. 1.
HD, MU, NT, are* equal: therefore three planes of * 24. 11.
the solid LP are equal and similar to three planes of the solid KR, as also to three planes of the solid AV: but the three planes opposite to these three are equal and similar* to them in the several solids, and none of * 24. 11.
their solid angles are contained by more than three plane angles: therefore the three solids LP, KR, AV are equal* to one another. For the same reason, the * C. 11.
three solids ED, HU, MT are equal to one another: therefore what multiple soever the base LF is of the base AF, the same multiple is the solid LV of the solid AV. For the same reason, whatever multiple the base NF is of the base HF, the same multiple is the solid NV of the solid ED: and if the base LF be equal to the base NF, the solid LV is equal* to the solid NV; * C. 11.
if the base LF be greater than the base NF, the solid LV is greater than the solid NV; and if less, less. Since then there are four magnitudes, viz. the two bases AF, FH, and the two solids AV, ED, and of the base AF and solid AV, the base LF and solid LV are any equimultiples whatever; and of the base FH and solid ED, the base FN and solid NV are any equimultiples whatever; and it has been proved, that if the base LF is greater than the base FN, the solid LV is greater than the solid NV; and if equal, equal; and if less, less: therefore* as the base AF is to the base * 5 Def. 5.
FH, so is the solid AV to the solid ED. Wherefore, if a parallelopiped, &c. Q. E. D.

PROP. XXVI. PROB.

At a given point in a given straight line, to make a solid angle equal to a given solid angle contained by three plane angles.

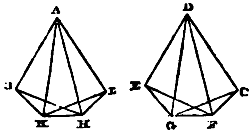
Let AB be a given straight line, A a given point in it, and D a given solid angle contained by the three plane angles EDC, EDF, FDC: it is required to make at the point A, in the straight line AB, a solid angle equal to the solid angle D.

In the straight line DF take any point F, from
* 11. 11. which draw* FG perpendicular to the plane EDC, meeting that plane in G; join DG, and at the point A
* 23. 1. in the straight line AB make* the angle BAL equal to the angle EDC, and in the plane BAL make the angle BAK equal to the angle EDG; then make AK
* 12. 11. equal to DG, and from the point K erect* KH at right angles to the plane BAL; and make KH equal to GF, and join AH: then the solid angle at A, which is contained by the three plane angles BAL, BAH, HAL, shall be equal to the solid angle at D contained by the three plane angles EDC, EDF, FDC.

Take the equal straight lines AB, DE, and join HB, KB, FE, GE. And because FG is perpendicular to
* 3 Def. 11. the plane EDC, it makes right angles* with every straight line meeting it in that plane: therefore each of the angles FGD, FGE, is a right angle. For the same reason, HKA, HKB are right angles: and because KA, AB are equal to GD, DE, each to each, and contain equal angles, therefore the base BK is
* 4. 1. equal* to the base EG: and KH is equal to GF, and
* 4. 1. HKB, FGE, are right angles, therefore HB is* equal to FE. Again, because AK, KH are equal to DG, GF, each to each, and contain right angles, the base AH is equal to the base DF; and AB is equal to DE:

therefore HA, AB are equal to FD, DE, each to each, and the base HB is equal to the base FE; therefore the angle BAH is equal* to the angle EDF. For the same reason, the angle HAL is equal to the angle FDC: because if AL and DC be made equal, and KL, HL, GC, FC be joined, since the whole angle BAL is equal to the whole EDC, and the parts of them BAK, EDG are, by the construction, equal; therefore the remaining angle KAL is equal to the remaining angle GDC: and because KA, AL are equal to GD, DC, each to each, and contain equal angles, the base KL is equal* to the base GC: and KH is equal to GF, so that LK, KH are equal to CG, GF, each to each, and they contain right* angles; therefore the base HL is equal* to the base FC. Again, because HA, AL are equal to FD, DC, each to each, and the base HL to the base FC, the angle HAL is equal* to the angle FDC. Therefore, because the three plane angles BAL, BAH, HAL, which contain the solid angle at A, are equal to the three plane angles EDC, EDF, FDC, which contain the solid angle at D, each to each, and are situated in the same order, the solid angle at A is equal* to the solid angle at D. Therefore, at a given point in a given straight line, a solid angle has been made equal to a given solid angle contained by three plane angles. Which was to be done.

* 8. 1.
* 4. 1.
* 3 Def. 11.
* 4. 1.
* 8. 1.
* B. 11.



PROP. XXVII. PROB.

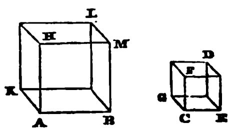
To describe, from a given straight line, a parallelopiped similar, and similarly situated to one given.

Let AB be the given straight line, and CD the given

parallelopiped: it is required from AB, to describe a parallelopiped similar and similarly situated to CD.

* 26. 11. At the point A of the given straight line AB, make* a solid angle equal to the solid angle at C, and let BAK, KAH, HAB be the three plane angles which contain it, so that BAK be equal to the angle ECG, and KAH to GCF, and HAB to FCE: and as EC to
* 12. 6. CG, so make* BA to AK; and as GC to CF, so make*
* 12. 6. * 22. 5. KA to AH; wherefore, ex æquali,* as EC to CF, so is BA to AH. Complete the parallelogram BH, and the solid AL: then AL shall be similar and similarly situated to CD.

Because, as EC to CG, so BA to AK, the sides about the equal angles ECG, BAK are proportionals; there-
* 1 Def. 6. fore the parallelogram BK is similar* to EG. For the same reason, the parallelogram KH is similar to GF, and HB to FE: wherefore three parallelograms of the solid AL are similar to three of the solid CD; and the

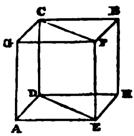


* 24. 11. three opposite ones in each solid are equal* and similar to these, each to each. Also, because the plane angles which contain the solid angles of the figures are equal, each to each, and situated in the
* B. 11. same order, the solid angles are equal* each to each:
* 11 Def. 11. therefore the solid AL is similar* to the solid CD. Wherefore from a given straight line AB, a parallelopiped AL has been described similar and similarly situated to the given one CD. Which was to be done.

PROP. XXVIII. THEOR.

If a parallelopiped be cut by a plane passing through the diagonals of two of the opposite planes, it shall be cut in two equal parts.

Let AB be a parallelopiped, and DE, CF the diagonals of the opposite parallelograms AH, GB, viz. those which are drawn betwixt the equal angles in each: and because CD, FE are each of them parallel to GA, and not in the same plane with it, CD, FE are* parallel; * 9. 11.
wherefore the diagonals CF, DE are in the plane in which the parallels are, and are themselves* parallels: * 16. 11.
then the plane CDEF shall cut the solid AB into two equal parts.



Because the triangle CGF is equal* * 34. 1.
to the triangle CBF, and the triangle DAE to DHE; and that the parallelogram CA is equal* and similar * 24. 11.
to the opposite one BE; and the parallelogram GE to CH: therefore the prism contained by the two triangles CGF, DAE, and the three parallelograms CA, GE, EC, is equal* to the prism contained by the two * C. 11.
triangles CBF, DHE, and the three parallelograms BE, CH, EC; because they are contained by the same number of equal and similar planes, alike situated, and none of their solid angles are contained by more than three plane angles. Therefore the solid AB is cut into two equal parts by the plane CDEF. Q. E. D.

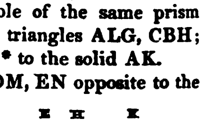
'N. B. The insisting straight lines of a parallelopiped, mentioned in the next and some following propositions, are the sides of the parallelograms betwixt the base and the opposite plane parallel to it.'

PROP. XXIX. THEOR.

Parallelopipeds upon the same base, and of the same altitude, the insisting straight lines of which are terminated in the same straight lines in the plane opposite to the base, are equal to one another.

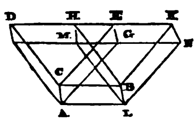
Let the parallelopipeds AH, AK be upon the same base AB, and of the same altitude, and let their insisting straight lines AF, AG, LM, LN, be terminated in the same straight line FN, and CD, CE, BH, BK be terminated in the same straight line DK; the solid AH shall be equal to the solid AK.

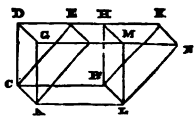
First, let the parallelograms DG, HN, which are opposite to the base AB, have a common side HG. Then, because the solid AH is cut by the plane AGHC passing through the diagonals AG, CH of the opposite
* 28. 11. planes ALGF, CBHD, AH is cut into two equal parts* by the plane AGHC: therefore the solid AH is double of the prism which is contained betwixt the triangles ALG, CBH. For the same reason, because the solid AK is cut by the plane LGHB through the diagonals LG, BH of the opposite planes ALNG, CBKH, the solid AK is double of the same prism which is contained betwixt the triangles ALG, CBH;



* 6 Ax. therefore the solid AH is equal* to the solid AK.

But, let the parallelograms DM, EN opposite to the





base, have no common side. Then, because CH, CK

are parallelograms, CB is equal* to each of the opposite sides DH, EK; wherefore DH is equal to EK: add, or take away the common part HE; then* DE is equal to HK: wherefore the triangle CDE is equal* to the triangle BHK: and the parallelogram DG is equal* to the parallelogram HN. For the same reason, the triangle AFG is equal to the triangle LMN, and the parallelogram CF is equal* to the parallelogram BM, and CG to BN; for they are opposite. Therefore the prism which is contained by the two triangles AFG, CDE, and the three parallelograms AD, DG, GC is equal* to the prism contained by the two triangles LMN, BHK, and the three parallelograms BM, MK, KL. Therefore if the prism LMNBHK be taken from the solid of which the base is the parallelogram AB, and in which FDKN is the one opposite to it; and if from this same solid there be taken the prism AFGCDE; the remaining solid, viz. the parallelopiped AH, is equal* to the remaining parallelopiped AK. Therefore parallelopipeds, &c. Q. E. D.

* 34. 1.
* 2 or 3 Ax.
* 38. 1.
* 36. 1.
* 24. 11.
* C. 11.
* 3 Ax.

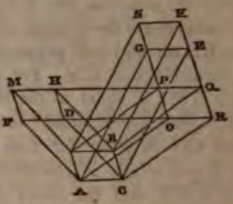
PROP. XXX. THEOR.

Parallelopipeds upon the same base, and of the same altitude, the insisting straight lines of which are not terminated in the same straight lines in the plane opposite to the base, are equal to one another.

Let the parallelopipeds CM, CN, be upon the same base AB, and of the same altitude, but their insisting straight lines AF, AG, LM, LN, CD, CE, BH, BK, not terminated in the same straight lines: the solids CM, CN shall be equal to one another.

Produce FD, MH, and NG, KE, and let them meet one another in the points O, P, Q, R; and join AO, LP, BQ, CR. And because the plane LBHM is pa-

rallel to the opposite plane ACDF, and that the plane LBHM is that in which are the parallels LB, MHPQ,



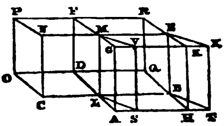
in which also is the figure BLPQ; and the plane ACDF is that in which are the parallels AC, FDOR, in which also is the figure CAOR; therefore the figures BLPQ, CAOR are in parallel planes. In like manner, because the plane ALNG is parallel to the opposite plane CBKE, and that the plane ALNG is that in which are the parallels AL, OPGN, in which also is the figure ALPO; and the plane CBKE is that in which are the parallels CB, RQEK, in which also is the figure CBQR; therefore the figures ALPO, CBQR are in parallel planes: and the planes ACBL, ORQP
* Hyp. are* parallel; therefore the solid CP is a parallelopiped;
* 29. 11. but the solid CM is equal* to the solid CP, because they are upon the same base ACBL, and their insisting straight lines AF, AO; CD, CR; LM, LP; BH, BQ are in the same straight lines FR, MQ: and the solid
* 29. 11. CP is equal* to the solid CN; for they are upon the same base ACBL, and their insisting straight lines AO, AG; LP, LN; CR, CE; BQ, BK are in the same straight lines ON, RK: therefore the solid CM is equal to the solid CN. Wherefore parallelopipeds, &c. Q. E. D.

PROP. XXXI. THEOR.

Parallelopipeds which are upon equal bases, and of the same altitude, are equal to one another.

Let the parallelopipeds AE, CF, be upon equal bases AB, CD, and be of the same altitude; the solid AE shall be equal to the solid CF.

First, let the insisting straight lines be at right angles to the bases AB, CD, and let the bases be placed in the same plane, and so that the sides CL, LB be in a straight line; therefore the straight line LM, which is at right angles to the plane in which the bases are, in the point L, is common* to the two solids AE, CF; * 13. 11.
let the other insisting lines of the solids be AG, HK, BE; DF, OP, CN: and first, let the angle ALB be equal to the angle CLD; then AL, LD are in a straight line.* Produce OD, HB, and let them meet * 14. 1.
in Q, and complete the parallelopiped LR, the base of which is the parallelogram LQ, and of which LM is



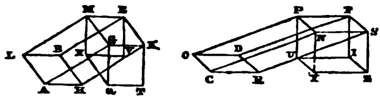
one of its insisting straight lines: therefore, because the parallelogram AB is equal to CD, as the base AB is to the base LQ, so is* the base CD to the same LQ. * 7. 5.
And because the parallelopiped AR is cut by the plane LMEB, which is parallel to the opposite planes AK, DR; as the base AB is to the base LQ, so is* the * 25. 11.
solid AE to the solid LR: for the same reason, because the *parallelopiped CR* is cut by the plane LMFD,

which is parallel to the opposite planes CP, BR; as the base CD to the base LQ, so is the solid CF to the solid LR: but as the base AB to the base LQ, so the base CD to the base LQ, as was before proved; there-
* 11. 5. fore * as the solid AE to the solid LR, so is the solid CF to the solid LR; and therefore the solid AE is
* 9. 5 equal * to the solid CF.

But let the parallelopipeds SE, CF be upon equal bases SB, CD, and be of the same altitude, and let their insisting straight lines be at right angles to the bases; and place the bases SB, CD in the same plane, so that CL, LB be in a straight line; and let the angles SLB, CLD be unequal; the solid SE shall be equal to the solid CF. Produce DL, TS until they meet in A, and from B draw BH parallel to DA; and let HB, OD produced meet in Q, and complete the
* 29. 11. solids AE, LR: therefore the solid AE is equal * to the solid SE; for they are upon the same base LE, and of the same altitude, and their insisting straight lines, viz. LA, LS, BH, BT; MG, MV, EK, EX are in the same straight lines AT, GX: and because the
* 35. 1. parallelogram AB is equal * to SB, for they are upon the same base LB, and between the same parallels LB, AT; and that the base SB is equal to the base CD; therefore the base AB is equal to the base CD, and the angle ALB is equal to the angle CLD: therefore, by the first case, the solid AE is equal to the solid CF; but the solid AE is equal to the solid SE, as was demonstrated; therefore the solid SE is equal to the solid CF.

But, if the insisting straight lines AG, HK, BE, LM; CN, RS, DF, OP, be not at right angles to the bases AB, CD: in this case likewise the solid AE shall be equal to the solid CF. From the points G, K, E, M; N, S, F, P, draw the straight lines GQ, KT, EV,

MX; NY, SZ, FI, PU, perpendicular* to the plane * 11. 11.
in which are the bases AB, CD; and let them meet it in the points Q, T, V, X; Y, Z, I, U, and join QT, TV, VX, XQ; YZ, ZI, IU, UY. Then, because GQ, KT, are at right angles to the same plane, they are



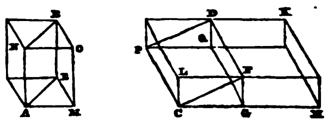
parallel* to one another: and MG, EK are parallels; * 6. 11.
therefore the planes MQ, ET, of which one passes through MG, GQ, and the other through EK, KT which are parallel to MG, GQ, and not in the same plane with them, are parallel* to one another: for the * 15. 11.
same reason, the planes MV, GT are parallel to one another; therefore the solid QE is a parallelopiped. In like manner, it may be proved, that the solid YF is a parallelopiped: but, from what has been demonstrated, the solid EQ is equal to the solid FY, because they are upon equal bases MK, PS, and of the same altitude, and have their insisting straight lines at right angles to the bases: and the solid EQ is equal* to the * 29 or
solid AE, because they are upon the same base and of 30. 11.
the same altitude: and for the same reason, the solid FY is equal to the solid CF: therefore the solid AE is equal to the solid CF: wherefore parallelopipeds, &c. Q. E. D.

PROP. XXXII. THEOR.

Parallelopipeds which have the same altitude, are to one another as their bases.

Let AB, CD be parallelopipeds of the same altitude; they are to one another as their bases; that is, as the base AE, to the base CF, so shall the solid AB be to the solid CD.

To the straight line FG apply the parallelogram FH
* Cor. 45. 1. equal* to AE, so that the angle FGH be equal to the angle LCG; and complete the parallelopiped GK upon the base FH, one of whose insisting lines is FD, whereby the solids CD, GK must be of the same alti-
* 31. 11. tude: therefore the solid AB is equal* to the solid GK; because they are upon equal bases AE, FH, and



are of the same altitude; and because the parallelopiped CK is cut by the plane DG which is parallel to
* 25. 11. its opposite planes, the base HF is* to the base FC, as the solid HD to the solid DC: but the base HF is equal to the base AE, and the solid GK to the solid AB: therefore, as the base AE to the base CF, so is the solid AB to the solid CD. Wherefore parallelopipeds, &c. Q. E. D.

Cor. From this it is manifest that prisms upon triangular bases, of the same altitude, are to one another as their bases.

Let the prisms, the bases of which are the triangles AEM, CFG, and NBO, PDQ the triangles opposite to them, have the same altitude; they shall be to one

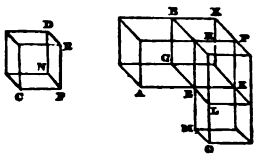
another as their bases. Complete the parallelograms AE, CF, and the parallelopipeds AB, CD, in the first of which let MO, and in the other let GQ be one of the insisting lines: and because the parallelopipeds AB, CD have the same altitude, they are to one another as the base AE is to the base CF; wherefore the prisms, which are their halves* are to one another as the base AE to the base CF; that is, as the triangle AEM to the triangle CFG. * 28. 11.

PROP. XXXIII. THEOR.

Similar parallelopipeds are one to another in the triplicate ratio of their homologous sides.

Let AB, CD be similar parallelopipeds, and the side AE homologous to the side CF: the solid AB shall have to the solid CD, the triplicate ratio of that which AE has to CF.

Produce AE, GE, HE, and in these produced take EK equal to CF, EL equal to FN, and EM equal to FR; and complete the parallelogram KL, and the solid KO. Because KE, EL are equal to CF, FN, each to each, and the angle KEL equal to the angle CFN, because it is equal to the angle AEG which is equal to CFN, by reason that the solids AB, CD are similar; therefore the parallelogram KL is similar and equal to the pa-



rallelogram CN. For the same reason, the parallelogram MK is similar and equal to CR, and also OE to

FD: therefore three parallelograms of the solid KO are equal and similar to three parallelograms of the solid CD: and the three opposite ones in each solid
* 24. 11. are equal* and similar to these: therefore the solid
* C. 11. KO is equal* and similar to the solid CD. Complete the parallelogram GK, and upon the bases GK, KL, complete the solids EX, LP, so that EH be an insisting straight line in each of them, whereby they must be of the same altitude with the solid AB. And because the solids AB, CD are similar, and, by permutation, as AE is to CF, so is EG to FN, and so is EH to FR; but FC is equal to EK, and FN to EL, and FR to EM: therefore, as AE to EK, so is EG to EL, and so is HE
* 1. 6. to EM: but, as AE to EK, so* is the parallelogram AG to the parallelogram GK; and as GE to EL, so
* 1. 6. is* GK to KL and as HE to EM, so* is PE to KM:
* 1. 6 therefore as the parallelogram AG to the parallelogram GK, so is GK to KL, and PE to KM: but as AG to
* 25. 11. GK, so* is the solid AB to the solid EX; and as GK
* 25. 11. to KL, so* is the solid EX to the solid PL; and as
* 25. 11. PE to KM, so* is the solid PL to the solid KO: therefore as the solid AB to the solid EX, so is EX to PL, and PL to KO. But if four magnitudes be continual proportionals, the first is said to have to the fourth the
* 11 Def. 5. triplicate* ratio of that which it has to the second: therefore the solid AB has to the solid KO, the triplicate ratio of that which AB has to EX: but as AB is to EX, so is the parallelogram AG to the parallelogram GK, and the straight line AE to the straight line EK: wherefore the solid AB has to the solid KO, the triplicate ratio of that which AE has to EK. But the solid KO is equal to the solid CD, and the straight line EK is equal to the straight line CF: therefore the solid AB has to the solid CD, the triplicate ratio of that which the side AE has to the homologous side CF. Therefore, similar parallelopipeds, &c. Q.E.D.

COR. From this it is manifest, that if four straight lines be continual proportionals, as the first is to the fourth, so is the parallelopiped described from the first to the similar solid similarly described from the second; because the first straight line has to the fourth the triplicate ratio of that which it has to the second.

PROP. D. THEOR.

Parallelopipeds contained by parallelograms equiangular to one another, each to each, that is, of which the solid angles are equal, each to each, have to one another the ratio which is the same with the ratio compounded of the ratios of their sides.

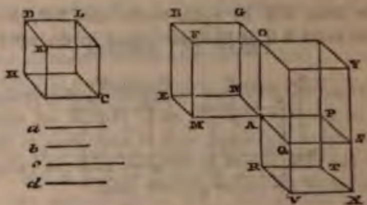
Let AB, CD be parallelopipeds, of which AB is contained by the parallelograms AE, AF, AG equiangular, each to each, to the parallelograms CH, CK, CL, which contain the solid CD: the ratio which the solid AB has to the solid CD shall be the same with that which is compounded of the ratios of the sides AM to DL, AN to DK, and AO to DH.

Produce MA, NA, OA to P, Q, R, so that AP be equal to DL, AQ to DK, and AR to DH: and complete the parallelopiped AX contained by the parallelograms AS, AT, AV similar and equal to CH, CK, CL, each to each: therefore the solid AX is equal* to the solid CD. Complete likewise the solid AY, the base of which is AS, and of which AO is one of its insisting straight lines. Take any straight line a, and as MA to AP, so make* a to b; and as NA to AQ, so make b to c: and as AO to AR, so c to d: then, because the parallelogram AE is equiangular to AS, AE is to AS, as the straight line a to c, as is demonstrated in the 23rd *Prop. Book* vi.; and the solids AB, AY, *being betwixt* the parallel planes BOY, EAS, are of

* C. 11.

* 12. 6.

the same altitude: therefore the solid AB is to the
* 32. 11. solid AY, as* the base AE to the base AS; that is, as the straight line a is to the straight line c. And the
* 25. 11. solid AY is to the solid AX, as* the base OQ is to the base QR; that is, as the straight line OA to AR; that is, as the straight line c to the straight line d. And



because the solid AB is to the solid AY, as a is to c, and the solid AY to the solid AX, as c is to d; therefore ex æquali, the solid AB is to the solid AX, or CD which is equal to it, as a is to d. But the ratio of a to
* Def. A. 5. d is said to be compounded* of the ratios of a to b, b to c, and c to d, which are the same with the ratios of the sides MA to AP, NA to AQ, and OA to AR, each to each; and the sides AP, AQ, AR are equal to the sides DL, DK, DH, each to each: therefore the solid AB has to the solid CD the ratio which is the same with that which is compounded of the ratios of the sides AM to DL, AN to DK, and AO to DH. Q. E. D.

PROP. XXXIV. THEOR.

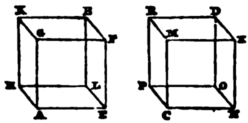
The bases and altitudes of equal parallelopipeds, are reciprocally proportional; and if the bases and altitudes be reciprocally proportional, the parallelopipeds are equal.

Let AB, CD be two parallelopipeds: and first, let

the insisting straight lines AG, EF, LB, HK; CM, NX, OD, PR be at right angles to the bases.

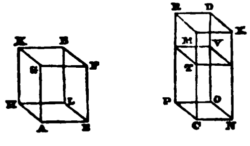
If the solid AB be equal to the solid CD, their bases shall be reciprocally proportional to their altitudes; that is, as the base EH is to the base NP, so shall CM be to AG.

If the base EH be equal to the base NP, then because the solid AB is likewise equal to the solid CD,



CM shall be equal to AG. Because if the bases EH, NP be equal, but the altitudes AG, CM be not equal, neither shall the solid AB be equal to the solid CD: but the solids are equal, by the hypothesis. Therefore the altitude CM is not unequal to the altitude AG: that is, it is equal to it. Wherefore as the base EH to the base NP, so is CM to AG.

Next, let the bases EH, NP be unequal, and let EH be the greater of the two: then since the solid AB is equal to the solid CD, CM is therefore greater than



AG. For, if it be not, neither also in this case would the solids AB, CD be equal, which, by the hypothesis, are equal. Make then CT equal to AG, and complete

the parallelopiped CV of which the base is NP, and altitude CT. Because the solid AB is equal to the solid CD, therefore the solid AB is to the solid CV,
* 7. 5. as* the solid CD to the solid CV. But as the solid
* 32. 11. AB to the solid CV, so* is the base EH to the base NP; for the solids AB, CV are of the same altitude;
* 25. 11. and as the solid CD to CV, so* is the base MP to the
* 1. 6. base PT, and so* is the straight line MC to CT; and CT is equal to AG: therefore, as the base EH to the base NP, so is MC to AG. Wherefore, the bases of the parallelopipeds AB, CD are reciprocally proportional to their altitudes.

Again, let the bases of the parallelopipeds AB, CD be reciprocally proportional to their altitudes; viz. as the base EH to the base NP, so let CM the altitude of the solid CD be to AG the altitude of the solid AB; the solid AB shall be equal to the solid CD. Then, if the base EH be equal to the base NP, since EH is to NP, as the altitude of the solid CD is to the altitude
* A. 5. of the solid AB, therefore the altitude of CD is equal* to the altitude of AB. But parallelopipeds upon
* 31. 11. equal bases, and of the same altitude, are equal* to one another: therefore the solid AB is equal to the solid CD.

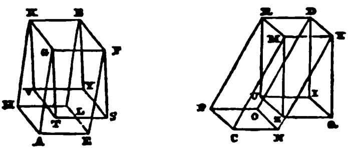
But let the bases EH, NP be unequal, and let EH be the greater of the two. Because as the base EH to the base NP, so is CM the altitude of the solid CD
* A. 5. to AG the altitude of AB; therefore CM is greater* than AG. Again, take CT equal to AG, and complete, as before, the solid CV. And, because the base EH is to the base NP, as CM to AG, and that AG is equal to CT, therefore the base EH is to the base NP, as
* 32. 11. MC to CT: but as the base EH is to NP, so* is the solid AB to the solid CV; for the solids AB, CV are
* 1. 6. of the same altitude; and as MC to CT, so* is the base

MP to the base PT, and the solid CD to the solid* CV: and therefore as the solid AB to the solid CV, so is the solid CD to the solid CV; that is, each of the solids AB, CD has the same ratio to the solid CV; and therefore the solid AB is equal* to the solid CD. * 25. 11. * 9. 5.

Second general case. Let the insisting straight lines FE, BL, GA, KH; XN, DO, MC, RP not be at right angles to the bases of the solids: in this case, likewise, if the solids AB, CD be equal, their bases are reciprocally proportional to their altitudes, viz. the base EH to the base NP, as the altitude of the solid CD to the altitude of the solid AB.

From the points F, B, K, G; X, D, R, M draw perpendiculars to the planes in which are the bases EH, NP meeting those planes in the points S, Y, V, T; Q, I, U, Z; and complete the solids FV, XU, which are parallelopipeds, as was proved in the last part of Prop. xxxi. of this Book.

Because the solid AB is equal to the solid CD, and that the solid BA is equal* to the solid BT, for they are upon the same base FK, and of the same altitude; and that the solid DC is equal* to the solid DZ, being * 29. or 30. 11. * 29. or 30. 11.



upon the same base XR, and of the same altitude; therefore the solid BT is equal to the solid DZ. But the bases are reciprocally proportional to the altitudes of equal parallelopipeds of which the insisting straight lines *are at right angles* to their bases, as before was

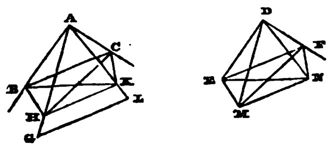
proved: therefore as the base FK to the base XR, so is the altitude of the solid DZ to the altitude of the solid BT: and the base FK is equal to the base EH, and the base XR to the base NP: therefore, as the base EH to the base NP, so is the altitude of the solid DZ to the altitude of the solid BT. But the altitudes of the solids DZ, DC, as also of the solids BT, BA are the same: therefore as the base EH to the base NP, so is the altitude of the solid CD to the altitude of the solid AB; that is, the bases of the parallelopipeds AB, CD are reciprocally proportional to their altitudes.

Next, let the bases of the solids AB, CD be reciprocally proportional to their altitudes, viz. the base EH to the base NP, as the altitude of the solid CD to the altitude of the solid AB; the solid AB shall be equal to the solid CD. The same construction being made; because, as the base EH to the base NP, so is the altitude of the solid CD to the altitude of the solid AB; and that the base EH is equal to the base FK; and NP to XR; therefore the base FK is to the base XR, as the altitude of the solid CD to the altitude of AB: but the altitudes of the solids AB, BT are the same, as also of CD and DZ; therefore as the base FK to the base XR, so is the altitude of the solid DZ to the altitude of the solid BT: therefore the bases of the solids BT, DZ are reciprocally proportional to their altitudes; and their insisting straight lines are at right angles to the bases; wherefore, as was before proved, the solid BT is equal to the solid DZ: but BT
* 29. or 30. 11. is equal* to the solid BA, and DZ to the solid DC, because they are upon the same bases, and of the same altitude. Therefore the solid AB is equal to the solid CD. Q. E. D.

PROP. XXXV. THEOR.

If, from the vertices of two equal plane angles, there be drawn two straight lines elevated above the planes in which the angles are, and containing equal angles with the sides of those angles, each to each; and if in the lines above the planes there be taken any points, and from them perpendiculars be drawn to the planes in which the first named angles are: and from the points in which they meet the planes, straight lines be drawn to the vertices of the angles first named; these straight lines shall contain equal angles with the straight lines which are above the planes of the angles.

Let BAC, EDF be two equal plane angles; and from the points A, D let the straight lines AG, DM be elevated above the planes of the angles, making equal angles with their sides each to each, viz. the angle GAB equal to the angle MDE, and GAC to MDF; and in AG, DM let any points G, M be taken, and from them let perpendiculars GL, MN be drawn* to * 11. 11.



the planes BAC, EDF meeting these planes in the points L, N; and join LA, ND: the angle GAL shall be equal to the angle MDN.

Make AH equal to DM, and through H draw HK parallel to GL: but GL is perpendicular to the plane

* 8. 11. BAC; wherefore HK is perpendicular* to the same plane. From the points K, N, to the straight lines AB, AC, DE, DF, draw perpendiculars KB, KC, NE, NF; and join HB, BC, ME, EF. Because HK is perpendicular to the plane BAC, the plane HBK which
* 18. 11. passes through HK is at right angles* to the plane BAC; and AB is drawn in the plane BAC at right angles to BK the common section of the two planes;
* 4 Def. 11. therefore AB is perpendicular* to the plane HBK,
* 3 Def. 11. and makes right angles* with every straight line meeting it in that plane: but BH meets it in that plane: therefore ABH is a right angle. For the same reason, DEM is a right angle, and is therefore equal
* Hyp. to the angle ABH: and the angle HAB is equal* to the angle MDE: therefore in the two triangles HAB, MDE there are two angles in the one equal to two angles in the other, each to each, and one side equal to one side, opposite to one of the equal angles in each, viz. HA equal to DM; therefore the remaining sides
* 26. 1. are equal* each to each: wherefore AB is equal to DE. In like manner, if HC and MF be joined, it may be demonstrated that AC is equal to DF: therefore, since AB is equal to DE, BA and AC are equal to ED and DF, each to each: and the angle BAC is
* Hyp. equal* to the angle EDF; wherefore the base BC is
* 4. 1. equal* to the base EF, and the remaining angles to the remaining angles: therefore the angle ABC is equal to the angle DEF: and the right angle ABK is equal to the right angle DEN, whence the remaining angle CBK is equal to the remaining angle FEN. For the same reason, the angle BCK is equal to the angle EFN: therefore in the two triangles BCK, EFN, there are two angles in the one equal to two angles in the other, each to each, and one side equal to one side adjacent to the equal angles in each, viz. BC equal to

EF; therefore the other sides are equal to the other sides; BK then is equal to EN: and AB is equal to DE; wherefore AB, BK are equal to DE, EN, each to each; and they contain right angles; wherefore the base AK is equal to the base DN. And since AH is equal to DM, the square of AH is equal to the square of DM: but the squares of AK, KH are equal to the square* of AH, because AKH is a right angle: and * 47. 1.
the squares of DN, NM are equal to the square of DM, for DNM is a right angle; wherefore the squares of AK, KH are equal to the squares of DN, NM; and of these the square of AK is equal to the square of DN: therefore the remaining square of KH is equal to the remaining square of NM; and the straight line KH to the straight line NM: and because HA, AK are equal to MD, DN each to each, and the base HK to the base MN as has been proved; therefore the angle HAK is equal* to the angle MDN: therefore, * 8. 1.
if from the vertices, &c. Q. E. D.

COR. From this it is manifest, that if, from the vertices of two equal plane angles, there be elevated two equal straight lines containing equal angles with the sides of the angles, each to each; the perpendiculars drawn from the extremities of the equal straight lines to the planes of the first angles are equal to one another.

Another Demonstration of the Corollary.

Let the plane angles BAC, EDF be equal to one another, and let AH, DM be two equal straight lines above the planes of the angles, containing equal angles with BA, AC; ED, DF, each to each, viz. the angle HAB equal to MDE, and HAC equal to the angle MDF; *and from H, M* let HK, MN be perpendicu-

lars to the planes BAC, EDF: HK shall be equal to MN.

Because the solid angle at A is contained by the three plane angles BAC, BAH, HAC, which are, each to each, equal to the three plane angles EDF, EDM, MDF containing the solid angle at D; the solid angles at A and D are equal, and therefore coincide with one another; to wit, if the plane angle BAC be applied to the plane angle EDF, the straight line AH coincides with DM, as was shown in Prop. B. of this Book. And because AH is equal to DM, the point H coincides with the point M: wherefore HK which is perpen-
* 13. 11. dicular to the plane BAC coincides with MN* which is perpendicular to the plane EDF, because these planes coincide with one another: therefore HK is equal to MN. Q. E. D.

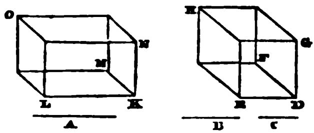
PROP. XXXVI. THEOR.

If three straight lines be proportionals, the parallelopiped described from all three as its sides, is equal to the equilateral parallelopiped described from the mean proportional, one of the solid angles of which is contained by three plane angles equal, each to each, to the three plane angles containing one of the solid angles of the other figure.

Let A, B, C be three proportionals, viz. A to B, as B to C: the solid described from A, B, C shall be equal to the equilateral solid described from B, equiangular to the other.

Take a solid angle D contained by three plane angles EDF, FDG, GDE; and make each of the straight lines ED, DF, DG equal to B, and complete the parallelopiped DH: make LK equal to A, and at

the point K in the straight line LK make* a solid angle contained by the three plane angles LKM, MKN, NKL equal to the angles EDF, FDG, GDE, each to each; and make KN equal to B, and KM equal to C; and complete the parallelopiped KO. And because, as A is to B, so is B to C, and that A is equal to LK, and B to each of the straight lines DE, DF, and C to KM; therefore LK is to ED, as DF to * 26. 11.



KM; that is, the sides about the equal angles are reciprocally proportional; therefore the parallelogram LM is equal* to EF. And because EDF, LKM are two equal plane angles, and the two equal straight lines DG, KN are drawn from their vertices above their planes, and contain equal angles with their sides; therefore the perpendiculars from the points G, N, to the planes EDF, LKM are equal* to one another: therefore the solids KO, DH are of the same altitude; and they are upon equal bases LM, EF, and therefore they are equal* to one another: but the solid KO is described from the three straight lines A, B, C, and the solid DH from the straight line B. Therefore if three straight lines, &c. Q. E. D.

* 14. 6.

*Cor. 35. 11.

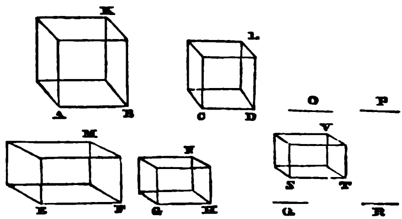
* 31. 11.

PROP. XXXVII. THEOR.

If four straight lines be proportionals, the similar parallelopipeds similarly described from them shall also be proportionals. And if the similar parallelopipeds similarly described from four straight lines be proportionals, the straight lines shall be proportionals.

Let the four straight lines AB, CD, EF, GH be proportionals, viz. as AB to CD, so EF to GH; and let the similar parallelopipeds AK, CL, EM, GN be similarly described from them: then AK shall be to CL, as EM to GN.

* 11. 6. Make* AB, CD, O, P continual proportionals, as also EF, GH, Q, R. And because as AB is to CD, so



* 11. 5. EF to GH; and that CD is* to O, as GH to Q, and
* 22. 5. O to P, as Q to R; therefore, ex æquali,* AB is to P,
* Cor. 33. 11. as EF to R: but as AB to P, so* is the solid AK to the solid CL; and as EF to R, so* is the solid EM to
* Cor. 33. 11. the solid GN: therefore* as the solid AK to the solid
* 11. 5. CL, so is the solid EM to the solid GN.

Next, let the solid AK be to the solid CL, as the

solid EM to the solid GN: the straight line AB shall be to CD, as EF to GH.

Take AB to CD, as EF to ST, and from ST describe* a parallelopiped SV similar and similarly situated to either of the solids EM, GN. And because AB is to CD, as EF to ST, and that from AB, CD the parallelopipeds AK, CL are similarly described; and in like manner the solids EM, SV from the straight lines EF, ST; therefore AK is to CL, as EM to SV: but, by the hypothesis, AK is to CL, as EM to GN: therefore GN is equal* to SV: but it is likewise similar and similarly situated to SV; therefore the planes which contain the solids GN, SV are similar and equal, and their homologous sides GH, ST equal to one another: and because as AB to CD, so EF to ST, and that ST is equal to GH; therefore AB is to CD, as EF to GH. Therefore, if four straight lines, &c. Q. E. D.

* 27. 11.

* 9. 5.

PROP. XXXVIII. THEOR.

" *If a plane be perpendicular to another plane, and a*
" *straight line be drawn from a point in one of*
" *the planes perpendicular to the other plane, this*
" *straight line shall fall on the common section of the*
" *planes.*"

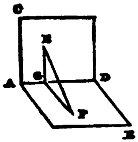
" Let the plane CD be perpendicular to the plane
" AB, and let AD be their common section; if any
" point E be taken in the plane CD, the perpendicular
" drawn from E to the plane AB shall fall on AD.

" For, if it does not, let it, if possible, fall elsewhere,
" as EF: and let it meet the plane AB in the point
" F; and from F draw,* in the plane AB a perpendi-
" cular *FG to DA*, which is also perpendicular* to the

* 12. 1.

* 4 Def. 11.

"plane CD; and join EG. Then because FG is perpendicular to the plane CD, and the straight line EG, which is in that plane, meets it; therefore FGE



* 3 Def. 11. "is a right* angle: but EF is also at right angles to the plane AB: and therefore EFG is a right angle; wherefore two of the angles of the triangle EFG are

* 17. 1. "equal together to two right angles; which is* absurd: therefore the perpendicular from the point E to the plane AB, does not fall elsewhere than upon the straight line AD: it therefore falls upon it. Therefore if a plane, &c. Q. E. D."

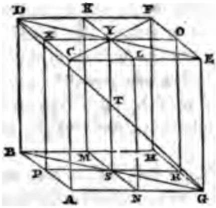
PROP. XXXIX. THEOR.

In a parallelopiped, if the sides of two of the opposite planes be divided each into two equal parts, the common section of the planes passing through the points of division, and the diameter of the parallelopiped cut each other into two equal parts.

Let the sides of the opposite planes CF, AH of the parallelopiped AF, be divided each into two equal parts in the points K, L, M, N; X, O, P, R; and join KL, MN, XO, PR. And because DK, CL are equal

* 33. 1. and parallel, KL is parallel* to DC: for the same reason, MN is parallel to BA: and BA is parallel to DC; therefore, because KL, BA are each of them parallel to DC, and not in the same plane with it, KL

is parallel* to BA. And because KL, MN are each * 9. 11.
of them parallel to BA, and not in the same plane



with it, KL is parallel* to MN; wherefore KL, MN * 9. 11.
are in one plane. In like manner, it may be proved, that XO, PR are in one plane. Let YS be the common section of the planes KN, XR; and DG the diameter of the parallelopiped AF: YS and DG shall meet, and cut one another into two equal parts.

Join DY, YE, BS, SG. Because DX is parallel to OE, the alternate angles DXY, YOE are equal* to one * 29. 1.
another: and because DX, XY are equal to EO, OY, each to each, and contain equal angles, the base DY is equal* to the base YE, and the other angles are equal; * 4. 1.
therefore the angle XYD is equal to the angle OYE, and DYE is a straight* line. For the same reason * 14. 1.
BSG is a straight line, and BS equal to SG. And because CA is equal and parallel to DB, and also equal and parallel to EG; therefore DB is equal and parallel* * 9. 11.
to EG: and DE, BG join their extremities; therefore DE is equal and parallel* to BG: and DG, YS are * 33. 1.
drawn from points in the one, to points in the other; and are therefore in one plane: whence it is manifest, that DG, YS must meet one another; let them meet in T. And because DE is parallel to BG, the alter-

* 29. 1. nate angles EDT, BGT are* equal; and the angle
* 15. 1. DTY is equal* to the angle GTS: therefore in the triangles DTY, GTS there are two angles in the one equal to two angles in the other, and one side equal to one side, opposite to two of the equal angles, viz. DY to GS; for they are the halves of DE, BG; therefore
* 26. 1. the remaining sides are equal,* each to each: therefore DT is equal to TG, and YT equal to TS. Wherefore, in a parallelopiped, &c. Q. E. D.

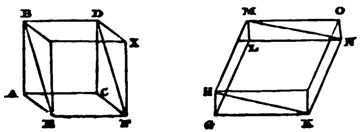
PROP. XL. THEOR.

If there be two triangular prisms of the same altitude, the base of one of which is a parallelogram, and the base of the other a triangle; if the parallelogram be double of the triangle, the prisms shall be equal to one another.

Let the prisms ABCDEF, GHKLMN be of the same altitude, the first whereof is contained by the two triangles ABE, CDF, and the three parallelograms AD, DE, EC; and the other by the two triangles GHK, LMN and the three parallelograms LH, HN, NG; and let one of them have a parallelogram AF, and the other a triangle GHK for its base; if the parallelogram AF be double of the triangle GHK, the prism ABCDEF shall be equal to the prism GHKLMN.

Complete the solids AX, GO; and because the parallelogram AF is double of the triangle GHK; and
* 34. 1. the parallelogram HK double* of the same triangle; therefore the parallelogram AF is equal to HK. But parallelopipeds upon equal bases, and of the same

titude, are equal* to one another: therefore the * 31. 11.
lid AX is equal to the solid GO; and the prism



BCDEF is half* of the solid AX; and the prism * 28. 11.
HKLMN half* of the solid GO: therefore the prism * 28. 11.
BCDEF is equal to the prism GHKLMN. Wherere, if there be two, &c. Q. E. D.

THE

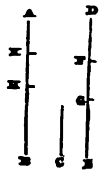
ELEMENTS OF EUCLID.

BOOK XII.

LEMMA I.[a]

If from the greater of two unequal magnitudes, there be taken more than its half, and from the remainder more than its half; and so on: there shall at length remain a magnitude less than the least of the proposed magnitudes.

Let AB and C be two unequal magnitudes, of which AB is the greater. If from AB there be taken more than its half, and from the remainder more than its half, and so on; there shall at length remain a magnitude less than C.



For C may be multiplied so as at length to become greater than AB. Let it be so multiplied, and let DE its multiple be greater than AB, and let DE be divided into DF, FG, GE, each equal to C. From

[a] *This is the first Proposition of the tenth Book, and is necessary to some of the Propositions of this Book.*

AB take BH greater than its half, and from the remainder AH take HK greater than its half, and so on until there be as many divisions in AB as there are in DE: and let the divisions in AB be AK, KH, HB; and the divisions in DE be DF, FG, GE. And because DE is greater than AB, and that EG taken from DE is not greater than its half, but BH taken from AB is greater than its half; therefore the remainder GD is greater than the remainder HA. Again, because GD is greater than HA, and that GF is not greater than the half of GD, but HK is greater than the half of HA; therefore the remainder FD is greater than the remainder AK: and FD is equal to C, therefore C is greater than AK: that is, AK is less than C. Q. E. D.

And if only the halves be taken away, the same thing may in the same way be demonstrated.

PROP. I. THEOR.

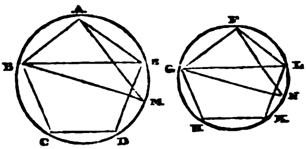
Similar polygons inscribed in circles, are to one another as the squares of their diameters.

Let ABCDE, FGHKL be two circles, and in them the similar polygons ABCDE, FGHKL; and let BM, GN be the diameters of the circles: the polygon ABCDE shall be to the polygon FGHKL, as the square of BM is to the square of GN.

Join BE, AM, GL, FN. And because the polygon ABCDE is similar to the polygon FGHKL, and because similar polygons may be divided into the same number of similar triangles; the triangles ABE, FGL, are similar and equiangular;* and therefore the angle AEB is equal to the angle FLG: but AEB is equal* to AMB, because they stand upon the same circumference; and the angle FLG is for the same reason,

* 20. 6.
* 21. 3.

equal to the angle FNG: therefore also the angle AMB is equal to FNG: and the right angle BAM is
* 31. 3. equal to the right* angle GFN; wherefore the remaining angles in the triangles ABM, FGN are equal,



and they are equiangular to one another: therefore as
* 4. 6. BM to GN, so* is BA to GF; and therefore the
* 10 Def. duplicate ratio of BM to GN, is the same* with the
5. & 22. 5. duplicate ratio of BA to GF. But the ratio of the
* 20. 6. square of BM to the square of GN, is the duplicate* ratio of that which BM has to GN; and the ratio of the polygon ABCDE to the polygon FGHKL is the
* 20. 6. duplicate* of that which BA has to GF: therefore as the polygon ABCDE to the polygon FGHKL, so is the square of BM to the square of GN. Wherefore similar polygons, &c. Q. E. D.

PROP. II. THEOR.

Circles are to one another as the squares of their diameters.

Let ABCD, EFGH be two circles, and BD, FH their diameters: as the square of BD to the square of FH, so shall the circle ABCD be to the circle EFGH.

For, if it be not so, the square of BD must be to the square of FH, as the circle ABCD is to some space either less than the circle EFGH, or greater

than it.[a] First, if possible, let it be to a space S less than the circle EFGH; and in the circle EFGH inscribe* the square EFGH: this square is greater than half of the circle EFGH; because if, through the points E, F, G, H, there be drawn tangents to the circle, the square EFGH is half* of the square described about the circle; and the circle is less than the square described about it; therefore the square EFGH is greater than half of the circle. Divide the circumferences EF, FG, GH, HE, each into two equal parts in the points K, L, M, N, and join EK, KF, FL, LG, GM, MH, HN, NE: therefore each of the triangles EKF, FLG, GMH, HNE is greater than half of the segment of the circle in which it stands; because, if straight lines touching the circle be drawn through the points K, L, M, N, and parallelograms upon the straight lines EF, FG, GH, HE, be completed; each of the triangles EKF, FLG, GMH, HNE is the half* of the parallelogram in which it is: but every segment is less than the parallelogram in which it is: therefore each of the triangles EKF, FLG, GMH, HNE is greater than half the segment of the circle which contains it. Again, if the circumferences before named be divided each into two equal parts, and their extremities be joined by straight lines by continuing to do this, and there will at length remain segments of the circle which, together, are less than the excess of the circle EFGH above the space S: because, by the preceding *lemma*, if from the greater of two unequal

* 6. 4.

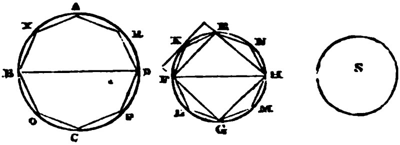
* 4. 1.

* 41. 1.

a For there is some square equal to the circle ABCD; let P be the side of it, and to three straight lines BD, FH, and P, there can be a fourth proportional; let this be Q: therefore* the squares of these four straight lines are proportionals; that is, to the squares of BD, FH, and the circle ABCD it is possible there may be a fourth proportional. Let this be S. And in like manner are to be understood some things in some of the following Propositions.

* 22. 6.

magnitudes there be taken more than its half, and from



the remainder more than its half, and so on, there at length remains a magnitude less than the least of the proposed magnitudes. Let then the segments EK, KF, FL, LG, GM, MH, HN, NE be those that remain and are together less than the excess of the circle EFGH above S: therefore the rest of the circle, viz. the polygon EKFLGMHN, is greater than the space S; that is, the space S is less than the polygon EKFLGMHN. Describe likewise in the circle ABCD the polygon AXBOCPDR similar to the polygon EKFLGMHN: therefore as the square of BD is to

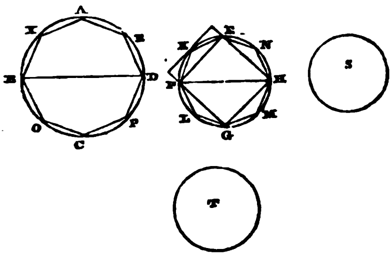
* 1. 12. the square of FH, so* is the polygon AXBOCPDR to the polygon EKFLGMHN: but the square of BD is

* Hyp. also to the square of FH,* as the circle ABCD is to the space S: therefore as the circle ABCD is to the space

* 11. 5. S, so is* the polygon AXBOCPDR to the polygon EKFLGMHN: but the circle ABCD is greater than the polygon contained in it; wherefore the space S is

* 14. 5. greater* than the polygon EKFLGMHN: but this is impossible, because it has been demonstrated to be less than the same polygon. Therefore the square of BD is not to the square of FH, as the circle ABCD is to any space less than the circle EFGH. In the same manner, it may be demonstrated, that neither is the square of FH to the square of BD, as the circle EFGH is to any space less than the circle ABCD. Nor is the square of BD to the square of FH, as the circle ABCD is to any space greater than the circle EFGH. For, if possible, let it be so to T a space greater than

the circle EFGH; therefore inversely as the square of FH to the square of BD, so is the space T to the



circle ABCD: but as the space[a] T is to the circle ABCD, so is the circle EFGH to some space, which must be less* than the circle ABCD, because the space T is greater, by hypothesis, than the circle EFGH: therefore as the square of FH is to the square of BD, so is the circle EFGH to a space less than the circle ABCD, which has been demonstrated to be impossible. Therefore the square of BD is not to the square of FH as the circle ABCD to any space greater than the circle EFGH: and it has been demonstrated, that neither is the square of BD to the square of FH, as the circle ABCD to any space less than the circle EFGH: therefore, as the square of BD to the square of FH, so is the circle ABCD to the circle EFGH.[b] Therefore, circles are, &c. Q. E. D. * 14. 5.

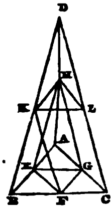
[a] For as in the foregoing note, it was explained how it was possible there could be a fourth proportional to the squares of BD, FH, and the circle ABCD, which was named S. So in like manner there can be a fourth proportional to this other space, named T, and the circles ABCD, EFGH. And the like is to be understood in some of the following Propositions.

[b] Because as a fourth proportional to the squares of BD, FH and the circle ABCD is possible, and that it can neither be less nor greater than the circle EFGH, it must be equal to it.

PROP. III. THEOR.

Every pyramid having a triangular base, may be divided into two equal and similar pyramids having triangular bases, and which are similar to the whole pyramid; and into two equal prisms which together are greater than half of the whole pyramid.

Let there be a pyramid of which the base is the triangle ABC and its vertex the point D: the pyramid ABCD may be divided into two equal and similar pyramids having triangular bases, and similar to the whole; and into two equal prisms which together shall be greater than half of the whole pyramid.



Divide AB, BC, CA, AD, DB, DC, each into two equal parts in the points E, F, G, H, K, L, and join EH, EG, GH, HK, KL, LH, EK, KF, FG. Because AE is equal to EB, and AH to HD, HE is
* 2. 6. parallel* to DB. For the same reason, HK is parallel to AB: therefore HEBK is a parallelogram, and HK
* 34. 1. equal* to EB: but EB is equal to AE; therefore also AE is equal to HK: and AH is equal to HD; wherefore EA, AH are equal to KH, HD, each to each; and
* 29. 1. the angle EAH is equal* to the angle KHD; therefore the base EH is equal to the base KD, and the
* 4. 1. triangle AEH equal* and similar to the triangle HKD. For the same reason, the triangle AGH is equal and similar to the triangle HLD. And because the two straight lines EH, HG which meet one another are parallel to KD, DL that meet one another, and are
* 10. 11. not in the same plane with them, they contain equal* angles; therefore the angle EHG is equal to the angle KDL. Again, because EH, HG, are equal to KD, DL, each to each, and the angle EHG equal to the angle

KDL; therefore the base EG is equal to the base KL: and the triangle EHG equal* and similar to the triangle KDL. * 4. 1. For the same reason, the triangle AEG is also equal and similar to the triangle HKL: therefore the pyramid of which the base is the triangle AEG, and of which the vertex is the point H, is equal* * C. 11. and similar to the pyramid the base of which is the triangle KHL, and vertex the point D. And because HK is parallel to AB a side of the triangle ADB, the triangle ADB is equiangular to the triangle HDK, and their sides are* proportionals: therefore the tri- * 4. 6. angle ADB is similar to the triangle HDK: and for the same reason, the triangle DBC is similar to the triangle DKL; and the triangle ADC to the triangle HDL; and also the triangle ABC to the triangle AEG. But the triangle AEG is similar to the triangle HKL, as before was proved; therefore the triangle ABC is similar* to the triangle HKL: and therefore the * 21. 6. pyramid of which the base is the triangle ABC, and vertex the point D, is similar* to the pyramid of which * B.11. & the base is the triangle HKL, and vertex the same 11 Def. 11. point D. But the pyramid of which the base is the triangle HKL, and vertex the point D, is similar, as has been proved, to the pyramid the base of which is the triangle AEG, and vertex the point H: wherefore the pyramid the base of which is the triangle ABC, and vertex the point D, is similar to the pyramid of which the base is the triangle AEG and vertex H: therefore each of the pyramids AEGH, HKLD is similar to the whole pyramid ABCD. And because BF is equal to FC, the parallelogram EBFG is double* of the triangle * 41. 1. GFC: but when there are two prisms of the same altitude, of which one has a parallelogram for its base, and the other a triangle that is half of the parallelogram, these prisms are equal* to one another; therefore * 40. 11. the prism having the parallelogram EBFG for its base,

and the straight line KH opposite to it, is equal to the prism having the triangle GFC for its base, and the triangle HKL opposite to it; for they are of the same
* 15. 11. altitude, because they are between the parallel* planes ABC, HKL. And it is manifest that each of these prisms is greater than either of the pyramids of which the triangles AEG, HKL are the bases, and the vertices the points H, D; because, if EF be joined, the prism having the parallelogram EBFG for its base, and KH the straight line opposite to it, is greater than the pyramid of which the base is the triangle EBF, and
* C. 11. vertex the point K; but this pyramid is equal* to the pyramid the base of which is the triangle AEG, and vertex the point H; because they are contained by equal and similar planes: wherefore the prism having the parallelogram EBFG for its base, and opposite side KH, is greater than the pyramid of which the base is the triangle AEG, and vertex the point H: and the prism of which the base is the parallelogram EBFG, and opposite side KH is equal to the prism having the triangle GFC for its base, and HKL the triangle opposite to it; and the pyramid of which the base is the triangle AEG, and vertex H, is equal to the pyramid of which the base is the triangle HKL, and vertex D: therefore the two prisms before mentioned are greater than the two pyramids of which the bases are the triangles AEG, HKL, and vertices the points H, D. Therefore the whole pyramid of which the base is the triangle ABC, and vertex the point D, is divided into two equal pyramids similar to one another, and to the whole pyramid; and into two equal prisms; and the two prisms are together greater than half of the whole pyramid. Q. E. D.

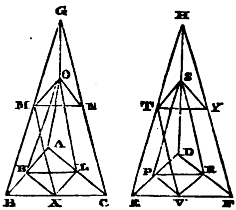
PROP. IV. THEOR.

If there be two pyramids of the same altitude upon triangular bases, and each of them be divided into two equal pyramids similar to the whole pyramid, and also into two equal prisms; and if each of these pyramids be divided in the same manner as the first two, and so on: as the base of one of the first two pyramids is to the base of the other, so are all the prisms in one of them to all the prisms in the other that are produced by the same number of divisions.

Let there be two pyramids of the same altitude upon the triangular bases ABC, DEF, and having their vertices in the points G, H; and let each of them be divided into two equal pyramids similar to the whole, and into two equal prisms; and let each of the pyramids thus made be conceived to be divided in the like manner, and so on: as the base ABC is to the base DEF, so shall all the prisms in the pyramid ABCG be to all the prisms in the pyramid DEFH made by the same number of divisions.

Make the same construction as in the foregoing Proposition. And because BX is equal to XC, and AL to LC, therefore XL is parallel* to AB, and the triangle ABC * 2. 6.
similar to the triangle LXC: for the same reason, the triangle DEF is similar to RVF: and because BC is double of CX, and EF double of FV, therefore* BC is * C. 5.
to CX, as EF to FV: and upon BC, CX are described the similar and similarly situated rectilineal figures ABC, LXC; and upon EF, FV, in like manner, are described the similar figures DEF, RVF: therefore, as the triangle ABC is to the triangle LXC, so* is the * 22. 6
triangle DEF to the triangle RVF, and, by permutation, as the triangle ABC to the triangle DEF, so is the triangle LXC to the triangle RVF. And because

the planes[a] ABC, OMN, as also the planes DEF, STY
* 15. 11. are parallel,* the perpendiculars drawn from the points G, H to the bases ABC, DEF, which, by the hypothesis, are equal to one another, shall be cut each
* 17. 11. into two equal* parts by the planes OMN, STY, because the straight lines GC, HF are cut into two equal parts in the points N, Y by the same planes: therefore the prisms LXCOMN, RVFSTY are of the same altitude; and therefore, as the base LXC to the base RVF; that is, as the triangle ABC to the triangle
* Cor. 32. 11. DEF, so* is the prism having the triangle LXC for its base, and OMN the triangle opposite to it, to the prism of which the base is the triangle RVF, and the opposite triangle STY. And because the two prisms in the pyramid ABCG are equal to one another, and also the two prisms in the pyramid DEFH equal to one another, as the prism of which the base is the parallelogram KBXL and opposite side MO, to the prism having the triangle LXC for its base, and OMN the triangle op-



* 7. 5. posite to it; so is the prism of which the base* is the parallelogram PEVR, and opposite side TS, to the prism of which the base is the triangle RVF, and opposite triangle STY: therefore, componendo, as the prisms KBXLMO LXCOMN together are to the

[a] Because GM is equal to MB, and GO to OA, therefore OM
* 2. 6. is parallel * to AB; in like manner ON is parallel to AC; there-
* 15. 11. fore the plane MON is parallel* to the plane BAC.

prism LXCOMN; so are the prisms PEVRTS, RVFSTY to the prism RVFSTY: and permutando, as the prisms KBXLMO, LXCOMN are to the prisms PEVRTS, RVFSTY; so is the prism LXCOMN to the prism RVFSTY: but as the prism LXCOMN to the prism RVFSTY, so is, as has been proved, the base ABC to the base DEF: therefore, as the base ABC to the base DEF, so are the two prisms in the pyramid ABCG to the two prisms in the pyramid DEFH. And likewise if the pyramids now made, for example, the two OMNG, STYH be divided in the same manner; as the base OMN is to the base STY, so are the two prisms in the pyramid OMNG to the two prisms in the pyramid STYH. But the base OMN is to the base STY, as the base ABC to the base DEF; therefore, as the base ABC to the base DEF, so are the two prisms in the pyramid ABCG to the two prisms in the pyramid DEFH; and so are the two prisms in the pyramid OMNG to the two prisms in the pyramid STYH; and so are all four to all four: and the same thing may be shown of the prisms made by dividing the pyramids AKLO and DPRS, and of all made by the same number of divisions. Q. E. D.

PROP. V. THEOR.

Pyramids of the same altitude which have triangular bases, are to one another as their bases.

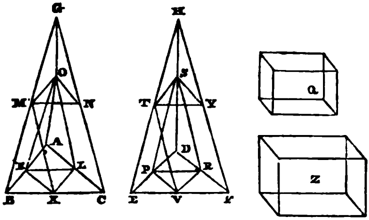
Let the pyramids of which the triangles ABC, DEF are the bases, and of which the vertices are the points G, H, be of the same altitude: as the base ABC to the base DEF, so shall the pyramid ABCG be to the pyramid DEFH.

For, if it be not so, the base ABC must be to the base DEF, as the pyramid ABCG to a solid either less than *the pyramid* DEFH, or greater than it.[a] First,

[a] See note at foot of page 283.

if possible, let it be to a solid less than it, viz. to the solid Q: and divide the pyramid DEFH into two equal pyramids, similar to the whole, and into two equal
* 3. 12. prisms: therefore these two prisms are greater* than the half of the whole pyramid. And again, let the pyramids made by this division be in like manner divided,
Lemma and so on, until the pyramids which remain undivided in the pyramid DEFH be, all of them together, less than the excess of the pyramid DEFH above the solid Q: let these, for example, be the pyramids DPRS, STYH: therefore the prisms, which make the rest of the pyramid DEFH, are greater than the solid Q; that is, the solid Q is less than these prisms. Divide likewise the pyramid ABCG in the same manner, and into as many parts, as the pyramid DEFH: therefore, as
* 4. 12. the base ABC to the base DEF, so* are the prisms in the pyramid ABCG to the prisms in the pyramid DEFH: but as the base ABC to the base DEF, so, by hypothesis, is the pyramid ABCG to the solid Q; and therefore, as the pyramid ABCG to the solid Q, so are the prisms in the pyramid ABCG to the prisms in the pyramid DEFH: but the pyramid ABCG is greater
* 14. 5. than the prisms contained in it; wherefore* also the solid Q is greater than the prisms in the pyramid DEFH: but this is impossible, because the solid Q has been demonstrated to be less than these prisms. Therefore the base ABC is not to the base DEF, as the pyramid ABCG to any solid which is less than the pyramid DEFH. In the same manner it may be demonstrated, that the base DEF is not to the base ABC, as the pyramid DEFH to any solid which is less than the pyramid ABCG. Nor can the base ABC be to the base DEF, as the pyramid ABCG to any solid which is greater than the pyramid DEFH. For if it be possible, let it be so to Z a solid greater than the pyramid. And because the base ABC is to the base DEF as the pyramid

ABCG to the solid Z; by inversion, as the base DEF to the base ABC, so is the solid Z to the pyramid



ABCG. But as the solid Z is to the pyramid ABCG, so is the pyramid DEFH to some solid[a] which must be less* than the pyramid ABCG, because the solid Z is greater than the pyramid DEFH. And therefore, as the base DEF to the base ABC, so is the pyramid DEFH to a solid less than the pyramid ABCG; the contrary to which has been proved: therefore the base ABC is not to the base DEF, as the pyramid ABCG to any solid which is greater than the pyramid DEFH. And it has been proved, that neither is the base ABC to the base DEF, as the pyramid ABCG to any solid which is less than the pyramid DEFH. Therefore, as the base ABC is to the base DEF, so is the pyramid ABCG to the pyramid DEFH. Wherefore pyramids, &c. Q. E. D. * 14. 5.

PROP. VI. THEOR.

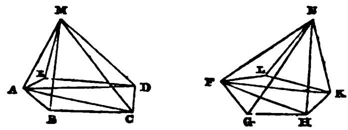
Pyramids of the same altitude which have polygons for their bases, are to one another as their bases.

Let the pyramids which have the polygons ABCDE, FGHKL for their bases, and their vertices in the points M, N, be of the same altitude: as the base ABCDE to

[a] See note at foot of page 283.

the base FGHKL, so shall the pyramid ABCDEM be to the pyramid FGHKLN.

Divide the base ABCDE into the triangles ABC, ACD, ADE; and the base FGHKL into the triangles FGH, FHK, FKL: and upon the bases ABC, ACD, ADE let there be as many pyramids of which the common vertex is the point M, and upon the remaining bases as many pyramids having their common vertex in the point N: therefore, since the triangle ABC is to
* 5. 12. the triangle FGH, as* the pyramid ABCM to the pyramid FGHN: and the triangle ACD to the triangle FGH, as the pyramid ACDM to the pyramid FGHN;



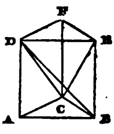
and also the triangle ADE to the triangle FGH, as the pyramid ADEM to the pyramid FGHN; as all the first
* 2 Cor. antecedents to their common consequent; so* are all
24. 5. the other antecedents to their common consequent; that is, as the base ABCDE to the base FGH, so is the pyramid ABCDEM to the pyramid FGHN. And, for the same reason, as the base FGHKL to the base FGH, so is the pyramid FGHKLN to the pyramid FGHN: and, by inversion, as the base FGH to the base FGHKL, so is the pyramid FGHN to the pyramid FGHKLN; then, because as the base ABCDE to the base FGH, so is the pyramid ABCDEM to the pyramid FGHN; and as the base FGH to the base FGHKL, so is the pyramid FGHN to the pyramid
* 22. 5. FGHKLN; therefore, ex æquali,* as the base ABCDE to the base FGHKL, so is the pyramid ABCDEM to the pyramid FGHKLN. Therefore pyramids, &c. Q. E. D.

PROP. VII. THEOR.

Every prism having a triangular base may be divided into three pyramids that have triangular bases, and are equal to one another.

Let there be a prism of which the base is the triangle ABC, and let DEF be the triangle opposite to it: the prism ABCDEF may be divided into three equal pyramids having triangular bases.

Join BD, EC, CD; and because ABED is a parallelogram of which BD is the diameter, the triangle ABD is equal* to the triangle EBD; therefore the pyramid of which the base is the triangle ABD, and vertex the point C, is equal* to the pyramid of which the base is the triangle EBD, and vertex the point C; but this pyramid is the same with the pyramid the base of which is the triangle EBC, and vertex the point D; for they are contained by the same planes: therefore the pyramid of which the base is the triangle ABD, and vertex the point C, is equal to the pyramid, the base of which is the triangle EBC, and vertex the point D. Again, because FCBE is a parallelogram of which the diameter is CE, the triangle ECF is equal* to the triangle ECB; therefore the pyramid of which the base is the triangle ECB, and vertex the point D, is equal to the pyramid, the base of which is the triangle ECF, and vertex the point D: but the pyramid of which the base is the triangle ECB, and vertex the point D has been proved equal to the pyramid of which the base is the triangle ABD, and vertex the point C: therefore the prism ABCDEF is divided into *three equal pyramids* having triangular bases, viz. into *the pyramids* ABDC, EBDC, ECFD. And because



* 34. 1.
* 5. 12.
* 34. 1.

the pyramid of which the base is the triangle ABD, and vertex the point C, is the same with the pyramid of which the base is the triangle ABC, and vertex the point D, for they are contained by the same planes; and that the pyramid of which the base is the triangle ABD, and vertex the point C, has been demonstrated to be a third part of the prism the base of which is the triangle ABC, and to which DEF is the opposite triangle; therefore the pyramid of which the base is the triangle ABC, and vertex the point D, is the third part of the prism which has the same base, viz. the triangle ABC, and DEF is the opposite triangle. Q. E. D.

COR. 1. From this it is manifest, that every pyramid is the third part of a prism which has the same base, and is ot an equal altitude with it; for if the base of the prism be any other figure than a triangle, it may be divided into prisms having triangular bases.

COR. 2. Prisms of equal altitudes are to one another as their bases; because the pyramids upon the same bases, and of the same altitude, are* to one another as their bases.

* 6. 12.

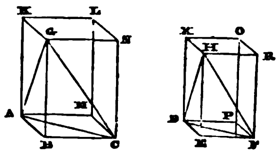
PROP. VIII. THEOR.

Similar pyramids having triangular bases are one to another in the triplicate ratio of that of their homologous sides.

Let the pyramids having the triangles ABC, DEF for their bases, and the points G, H for their vertices, be similar, and similarly situated; the pyramid ABCG shall have to the pyramid DEFH, the triplicate ratio of that which the side BC has to the homologous side EF.

Complete the parallelograms ABCM, GBCN, ABGK, and the parallelopiped BGML contained by these planes and those opposite to them: and, in like manner, complete the parallelopiped EHPO contained by

the three parallelograms DEFP, HEFR, DEHX, and those opposite to them. And because the pyramid ABCG is similar to the pyramid DEFH, the angle ABC is equal* to the angle DEF, and the angle GBC * 11 Def. 11.
to the angle HEF, and ABG to DEH: and AB is* * 1 Def. 6.
to BC, as DE to EF; that is, the sides about the equal



angles are proportionals; wherefore the parallelogram BM is similar to EP: for the same reason, the parallelogram BN is similar to ER, and BK to EX: therefore the three parallelograms BM, BN, BK are similar to the three EP, ER, EX: but the three BM, BN, BK, are equal and similar* to the three which are * 24. 11.
opposite to them, and the three EP, ER, EX equal and similar to the three opposite to them: wherefore the solids BGML, EHPO are contained by the same number of similar planes; and their solid angles* are * B. 11.
equal; and therefore the solid BGML is similar* to * 11 Def. 11.
the solid EHPO: but similar parallelopipeds have the triplicate* ratio of that which their homologous sides * 33. 11.
have: therefore the solid BGML has to the solid EHPO the triplicate ratio of that which the side BC has to the homologous side EF. But as the solid BGML is to the solid EHPO, so is* the pyramid * 15. 5.
ABCG to the pyramid DEFH; because the pyramids are the sixth part of the solids, since the prism, which is the half* of the parallelopiped, is triple* of the * 28. 11.
pyramid. Wherefore likewise the pyramid ABCG has * 7. 12.
to the pyramid DEFH, the triplicate ratio of that which BC has to the homologous side EF. Q. E. D.

COR. From this it is evident, that similar pyramids which have multangular bases, are likewise to one another in the triplicate ratio of their homologous sides. For they may be divided into similar pyramids having triangular bases, because the similar polygons, which are their bases, may be divided into the same number of similar triangles homologous to the whole polygons; therefore as one of the triangular pyramids in the first multangular pyramid is to one of the triangular pyramids in the other,* so are all the triangular pyramids in the first to all the triangular pyramids in the other: that is, so is the first multangular pyramid to the other: but one triangular pyramid is to its similar triangular pyramid, in the triplicate ratio of their homologous sides; and therefore the first multangular pyramid has to the other, the triplicate ratio of that which one of the sides of the first has to the homologous side of the other.

* 12. 5.

PROP. IX. THEOR.

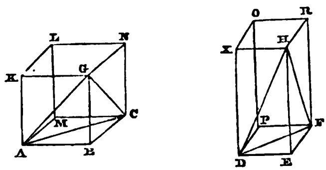
The bases and altitudes of equal pyramids having triangular bases are reciprocally proportional: and triangular pyramids of which the bases and altitudes are reciprocally proportional, are equal to one another.

Let the pyramids of which the triangles ABC, DEF are the bases, and which have their vertices in the points G, H, be equal to one another: the bases and altitudes of the pyramids ABCG, DEFH shall be reciprocally proportional, viz. the base ABC shall be to the base DEF, as the altitude of the pyramid DEFH to the altitude of the pyramid ABCG.

Complete the parallelograms AC, AG, GC, DF, DH, HF; and the parallelopipeds BGML, EHPO contained by these planes and those opposite to them.

And because the pyramid ABCG is equal to the pyramid DEFH, and that the solid BGML is sextuple* of the pyramid ABCG, and the solid EHPO sextuple of the pyramid DEFH; therefore the solid BGML is equal* to the solid EHPO. But the bases and altitudes of equal parallelopipeds are reciprocally proportional;* therefore as the base BM to the base EP, so is the altitude of the solid EHPO to the altitude of the solid BGML: but as the base BM to the base EP, so is* the triangle ABC to the triangle DEF; therefore as the triangle ABC to the triangle DEF, so is

* 28. 11. & 7. 12.
* 1 Ax. 5.
* 34. 11.
* 15. 5.



the altitude of the solid EHPO to the altitude of the solid BGML: but the altitude of the solid EHPO is the same with the altitude of the pyramid DEFH; and the altitude of the solid BGML is the same with the altitude of the pyramid ABCG: therefore, as the base ABC to the base DEF, so is the altitude of the pyramid DEFH to the altitude of the pyramid ABCG: therefore the bases and altitudes of the pyramids ABCG, DEFH are reciprocally proportional.

Again, let the bases and altitudes of the pyramids ABCG, DEFH be reciprocally proportional, viz. the base ABC to the base DEF, as the altitude of the pyramid DEFH to the altitude of the pyramid ABCG: the pyramid ABCG shall be equal to the pyramid DEFH.

The same construction being made, because as the *base ABC to the* base DEF, so is the altitude of the

pyramid DEFH to the altitude of the pyramid ABCG: and as the base ABC to the base DEF, so is the parallelogram BM to the parallelogram EP; therefore the parallelogram BM is to EP, as the altitude of the pyramid DEFH to the altitude of the pyramid ABCG: but the altitude of the pyramid DEFH is the same with the altitude of the parallelopiped EHPO; and the altitude of the pyramid ABCG is the same with the altitude of the parallelopiped BGML: therefore, as the base BM to the base EP, so is the altitude of the parallelopiped EHPO to the altitude of the parallelopiped BGML. But parallelopipeds having their bases and altitudes reciprocally proportional, are
* 34. 11. equal* to one another: therefore the parallelopiped BGML is equal to the parallelopiped EHPO. And the pyramid ABCG is the sixth part of the solid BGML, and the pyramid DEFH is the sixth part of the solid EHPO: therefore the pyramid ABCG is
* 2 A. 5. equal* to the pyramid DEFH. Therefore the bases, &c. Q. E. D.

PROP. X. THEOR.

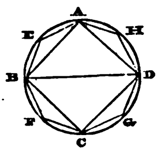
Every cone is the third part of a cylinder which has the same base, and is of an equal altitude with it.

Let a cone have the same base with a cylinder, viz. the circle ABCD, and the same altitude: the cone shall be the third part of the cylinder; that is, the cylinder shall be triple of the cone.

If the cylinder be not triple of the cone, it must either be greater than the triple, or less than it. First, if possible, let it be greater than the triple; and inscribe the square ABCD in the circle; this square is greater than the half of the circle ABCD.[a] Upon the

[a] As was shown in Prop. ii. of this Book.

square ABCD erect a prism of the same altitude with the cylinder; this prism shall be greater than half of the cylinder; because if a square is described about the circle, and a prism erected upon the square, of the same altitude with the cylinder, then the inscribed square is half of that circumscribed; and upon these square bases are erected parallelopipeds, viz. the prisms of the same altitude; therefore the prism upon the square ABCD is the half of the prism upon the square described about the circle: because they are to one another* as their bases: but the cylinder is less than the prism upon the square described about the circle ABCD: therefore the prism upon the square ABCD of the same altitude with the cylinder, is greater than half of the cylinder. Bisect the circumferences AB, BC, CD, DA in the points E, F, G, H; and join AE, EB, BF, FC, CG, GD, DH, HA: then, each of the triangles AEB, BFC, CGD, DHA is greater than the half of the segment of the circle in which it stands, as * 32. 11.

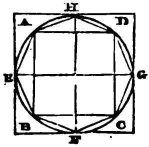


was proved in Prop. ii. of this Book. Erect prisms upon each of these triangles of the same altitude with the cylinder; each of these prisms shall be greater than half of the segment of the cylinder in which it is; because, if, through the points E, F, G, H, parallels be drawn to AB, BC, CD, DA, and parallelograms be completed upon the same AB, BC, CD, DA, and parallelopipeds be erected upon the parallelograms; the *prisms upon the* triangles AEB, BFC, CGD, DHA *are the halves of the* parallelopipeds:* and the seg- * 2 Cor. 7. 12.

ments of the cylinder which are upon the segments of the circle cut off by AB, BC, CD, DA, are less than the parallelopipeds which contain them; therefore the prisms upon the triangles AEB, BFC, CGD, DHA, are greater than half of the segments of the cylinder in which they are; therefore, if each of the circumferences be divided into two equal parts, and straight lines be drawn from the points of division to the extremities of the circumferences, and upon the triangles thus made, prisms be erected of the same altitude with the cylinder, and so on, there must at length remain some segments
Lemma of the cylinder which together are less than the excess of the cylinder above the triple of the cone. Let them be those upon the segments of the circle AE, EB, BF, FC, CG, GD, DH, HA: therefore the rest of the cylinder, that is, the prism of which the base is the polygon AEBFCGDH, and of which the altitude is the same with that of the cylinder, is greater than the
1 Cor. 7. 12. triple of the cone: but this prism is triple of the pyramid upon the same base, of which the vertex is the same with the vertex of the cone: therefore the pyramid upon the base AEBFCGDH, having the same vertex with the cone, is greater than the cone, of which the base is the circle ABCD: but this is impossible, because the pyramid is less than the cone, being contained within it; therefore the cylinder is not greater than the triple of the cone.

Nor can the cylinder be less than the triple of the cone. Let it be less, if possible: therefore, inversely, the cone is greater than the third part of the cylinder. In the circle ABCD inscribe a square; this square is greater than the half of the circle. Upon the square ABCD erect a pyramid having the same vertex with the cone; this pyramid is greater than the half of the cone: because as was before demonstrated, if a square be described about the circle, the square ABCD is the half

of it; and if, upon these squares there be erected parallelopipeds of the same altitude with the cone,



which are also prisms, the prism upon the square ABCD is the half of that which is upon the square described about the circle; for they are to one another as their bases;* as are also the third parts of them: * 32. 11.
therefore the pyramid, the base of which is the square ABCD, is half of the pyramid upon the square described about the circle. But this last pyramid is greater than the cone which it contains; therefore the pyramid upon the square ABCD, having the same vertex with the cone, is greater than the half of the cone. Bisect the circumferences AB, BC, CD, DA in the points E, F, G, H, and join AE, EB, BF, FC, CG, GD, DH, HA: therefore each of the triangles AEB, BFC, CGD, DHA is greater than half of the segment of the circle in which it is. Upon each of these triangles erect pyramids having the same vertex with the cone: therefore each of these pyramids is greater than the half of the segment of the cone in which it is, as before was demonstrated of the prisms and segments of the cylinder; and thus dividing each of the circumferences into two equal parts, and joining the points of division and their extremities by straight lines, and upon the triangles erecting pyramids having their vertices the same with that of the cone, and so on, there must at length remain some segments of the cone, *which together* are less* than the excess of the cone, *Lem.
above the third part of the cylinder. Let these be the

segments upon AE, EB, BF, FC, CG, GD, DH, HA: therefore the rest of the cone, that is, the pyramid, of which the base is the polygon AEBFCGDH, and of which the vertex is the same with that of the cone, is greater than the third part of the cylinder: but this pyramid is the third part of the prism upon the same base AEBFCGDH, and of the same altitude with the cylinder: therefore this prism is greater than the cylinder of which the base is the circle ABCD: but this is impossible, because the prism is less than the cylinder, being contained within it: therefore the cylinder is not less than the triple of the cone. And it has been demonstrated that neither is it greater than the triple: therefore the cylinder is triple of the cone, or, the cone is the third part of the cylinder. Wherefore every cone, &c. Q. E. D.

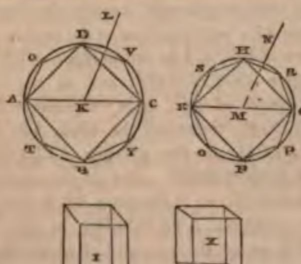
PROP. XI. THEOR.

Cones and cylinders of the same altitude, are to one another as their bases.

Let the cones and cylinders, of which the bases are the circles ABCD, EFGH, and the diameters of their bases AC, EG, and KL, MN the axes of the cones or cylinders, be of the same altitude: as the circle ABCD to the circle EFGH, so shall the cone AL be to the cone EN.

If it be not so, the circle ABCD must be to the circle EFGH, as the cone AL to some solid either less than the cone EN, or greater than it. First, if it be possible, let it be to a solid less than EN, viz. to the solid X; and let Z be the solid which is equal to the excess of the cone EN above the solid X; therefore the cone EN is equal to the solids X, Z together. In

the circle EFGH inscribe the square EFGH, therefore this square is greater than the half of the circle. Upon the square EFGH erect a pyramid of the same altitude with the cone; this pyramid shall be greater than half of the cone. For, if a square be described about the circle, and a pyramid be erected upon it, having the same vertex with the cone,[a] the pyramid inscribed in the cone is half of the pyramid circumscribed about it, because they are to one another as their bases:* but the cone is less than the circumscribed pyramid: therefore the pyramid of which the base is the square EFGH, and its vertex the same with that of the cone, is greater than half of the cone. Divide the circumferences EF, FG, GH, HE, each into two equal parts in the points O, P, R, S, and join EO, OF, FP, PG, GR, RH, HS, SE; therefore each of the triangles EOF, FPG, GRH, HSE is greater than half of the segment * 6. 12.



of the circle in which it is. Upon each of these tri-

[a] Vertex is put in place of altitude which is in the Greek, because the pyramid, in what follows, is supposed to be circumscribed about the cone, and so must have the same vertex. And the same change is made in some places following.

angles erect a pyramid having the same vertex with the cone; each of these pyramids is greater than the half of the segment of the cone in which it is: and thus dividing each of these circumferences into two equal parts, and from the points of division drawing straight lines to the extremities of the circumferences, and upon each of the triangles thus made erecting pyramids having the same vertex with the cone, and so on, there must at length remain some segments of the cone which
Lemma are together less than the solid Z; let these be the segments upon EO, OF, FP, PG, GR, RH, HS, SE: therefore the remainder of the cone, viz. the pyramid of which the base is the polygon EOFPGRHS, and its vertex the same with that of the cone, is greater than the solid X; that is, the solid X is less than the pyramid. In the circle ABCD inscribe the polygon ATBYCVDQ similar to the polygon EOFPGRHS, and upon it erect a pyramid having the same vertex with the cone AL: and because, as the square of AC is
* 1. 12. to the square of EG, so* is the polygon ATBYCVDQ to the polygon EOFPGRHS; and as the square of
* 2. 12. AC to the square of EG, so is* the circle ABCD to
* 11. 5. the circle EFGH; therefore the circle ABCD* is to the circle EFGH, as the polygon ATBYCVDQ to the polygon EOFPGRHS. But as the circle ABCD to the circle EFGH, so is the cone AL to the solid X; and as the polygon ATBYCVDQ to the polygon
* 6. 12. EOFPGRHS, so is* the pyramid of which the base is the first of these polygons, and vertex L, to the pyramid of which the base is the other polygon, and its vertex N: therefore, as the cone AL to the solid X, so is the pyramid of which the base is the polygon ATBYCVDQ, and vertex L, to the pyramid the base of which is the polygon EOFPGRHS, and vertex N. But the cone AL is greater than the pyramid con-
* 14. 5. tained in it; therefore the solid X is greater* than the

pyramid in the cone EN: but this is absurd, because it has been proved to be less than the pyramid. Therefore the circle ABCD is not to the circle EFGH, as the cone AL to any solid which is less than the cone EN. In the same manner it may be demonstrated that the circle EFGH is not to the circle ABCD, as the cone EN to any solid less than the cone AL. Nor can the circle ABCD be to the circle EFGH, as the cone AL to any solid greater than the cone EN. For, if it be possible, let it be so to I, a solid greater than the cone EN. Therefore, by inversion, as the circle EFGH to the circle ABCD, so is the solid I to the cone AL: but as the solid I to the cone AL, so is the cone EN to some solid, which must be less* than the cone AL, * 14. 5.
because the solid I is greater than the cone EN: therefore, as the circle EFGH is to the circle ABCD, so is the cone EN to a solid less than the cone AL, which was shown to be impossible: therefore the circle ABCD is not to the circle EFGH, as the cone AL is to any solid greater than the cone EN: and it has been demonstrated that neither is the circle ABCD to the circle EFGH, as the cone AL to any solid less than the cone EN: therefore the circle ABCD is to the circle EFGH, as the cone AL to the cone EN. But as the cone is to the cone, so* is the cylinder to the * 15. 5.
cylinder, because the cylinders are triple* of the cone * 10. 12.
each to each: therefore, as the circle ABCD to the circle EFGH, so are the cylinders upon them of the same altitude. Wherefore cones and cylinders of the same altitude are to one another as their bases. Q. E. D.

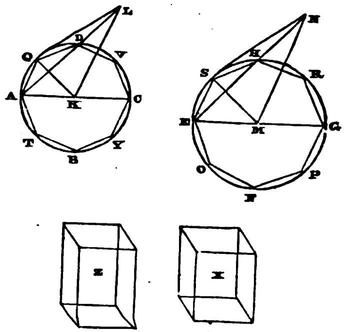
PROP. XII. THEOR.

Similar cones and cylinders have to one another the triplicate ratio of that which the diameters of their bases have.

Let the cones and cylinders of which the bases are

the circles ABCD, EFGH, and the diameters of the bases AC, EG, and KL, MN, the axes of the cones or cylinders, be similar: the cone, of which the base is the circle ABCD, and vertex the point L, shall have to the cone of which the base is the circle EFGH, and vertex N, the triplicate ratio of that which AC has to EG.

For if the cone ABCDL has not to the cone EFGHN the triplicate ratio of that which AC has to EG, the cone ABCDL shall have the triplicate of that ratio to some solid which is less or greater than the cone EFGHN. First, if possible, let it have it to a less, viz. to the solid X. Make the same construction as in the preceding proposition, and it may be demonstrated the very same way as in that proposition, that the solid X is less than the pyramid of which the base is the polygon EOFPGRHS, and vertex N. Inscribe also in the circle ABCD the polygon ATBYCVDQ similar to the polygon EOFPGRHS, upon which erect a pyramid having the same vertex with the cone; and let LAQ be



one of the triangles containing the pyramid upon the polygon ATBYCVDQ the vertex of which is L; and

let NES be one of the triangles containing the pyramid upon the polygon EOFPGRHS of which the vertex is N; and join KQ, MS: then because the cone ABCDL is similar to the cone EFGHN, AC is* to EG, as the axis KL to the axis MN; and as AC to EG, so* is AK to EM; therefore, as AK to EM, so is KL to MN; and, alternately, AK to KL, as EM to MN: and the right angles AKL, EMN are equal; therefore, the sides about these equal angles being proportionals, the triangle AKL is similar* to the triangle EMN. Again, because AK is to KQ, as EM to MS, and that these sides are about equal angles AKQ, EMS, because these angles are, each of them, the same part of four right angles at the centres K, M; therefore the triangle AKQ is similar* to the triangle EMS. And because it has been proved that as AK to KL, so is EM to MN, and that AK is equal to KQ; and EM to MS, therefore as QK to KL, so is SM to MN; and therefore, the sides about the right angles QKL, SMN being proportionals, the triangle LKQ is similar to the triangle NMS. And because of the similarity of the triangles AKL, EMN, as LA is to AK, so is NE to EM; and by the similarity of the triangles AKQ, EMS, as KA to AQ, so ME to ES; therefore ex æquali,* LA is to AQ, as NE to ES. Again, because of the similarity of the triangles LQK, NSM, as LQ to QK, so NS to SM: and from the similarity of the triangles KAQ, MES, as KQ to QA, so MS to SE; therefore ex æquali,* LQ is to QA, as NS to SE: and it was proved that QA is to AL, as SE to EN; therefore, again, ex æquali, as QL to LA, so is SN to NE: wherefore the triangles LQA, NSE, having the sides about all their angles proportionals, are equiangular* and similar to one another: and therefore the pyramid of which the base is the triangle AKQ, and vertex L, is similar to the pyramid the base of which is

* 24 Def. 11.
* 15. 5.
* 6. 6.
* 6. 6.
* 22. 5.
* 22 5.
* 5. 6.

the triangle EMS, and vertex N: because their solid
* B. 11. angles are equal* to one another, and they are contained by the same number of similar planes: but similar pyramids which have triangular bases have to
* 8. 12. one another the triplicate* ratio of that which their homologous sides have; therefore the pyramid AKQL has to the pyramid EMSN the triplicate ratio of that which AK has to EM. In the same manner, if straight lines be drawn from the points D, V, C, Y, B, T, to K, and from the points H, R, G, P, F, O to M, and pyramids be erected upon the triangles having the same vertices with the cones, it may be demonstrated that each pyramid in the first cone has to each in the other, taking them in the same order, the triplicate ratio of that which the side AK has to the side EM; that is, which AC has to EG; but as one antecedent to its consequent, so are all the antecedents to all the con-
* 12. 5. sequents;* therefore as the pyramid AKQL to the pyramid EMSN, so is the whole pyramid the base of which is the polygon DQATBYCV, and vertex L, to the whole pyramid of which the base is the polygon HSEOFPGR, and vertex N: wherefore also the first of these two last named pyramids has to the other the triplicate ratio of that which AC has to EG. But, by the hypothesis, the cone of which the base is the circle ABCD,and vertex L, has to the solid X, the triplicate ratio of that which AC has to EG: therefore, as the cone of which the base is the circle ABCD, and vertex L, is to the solid X, so is the pyramid the base of which is the polygon DQATBYCV, and vertex L, to the pyramid the base of which is the polygon HSEOFPGR and vertex N. But the said cone is greater than the pyramid contained in it, therefore the
* 14. 5. solid X is greater* than the pyramid, the base of which is the polygon HSEOFPGR, and vertex N; but this is impossible, because the solid X is less than the py-

ramid. Therefore the cone of which the base is the circle ABCD and vertex L, has not to any solid which is less than the cone of which the base is the circle EFGH and vertex N, the triplicate ratio of that which AC has to EG. In the same manner it may be demonstrated that neither has the cone EFGHN to any solid which is less than the cone ABCDL, the triplicate ratio of that which EG has to AC. Nor can the cone ABCDL have to any solid which is greater than the cone EFGHN, the triplicate ratio of that which AC has to EG. For, if it be possible, let it have it to a greater, viz. to the solid Z: therefore, inversely, the solid Z has to the cone ABCDL, the triplicate ratio of that which EG has to AC: but as the solid Z is to the cone ABCDL, so is the cone EFGHN to some solid, which must be less* than the cone ABCDL, because the solid Z is greater than the cone EFGHN: therefore the cone EFGHN has to a solid which is less than the cone ABCDL, the triplicate ratio of that which EG has to AC, which was demonstrated to be impossible. Therefore the cone ABCDL has not to any solid greater than the cone EFGHN, the triplicate ratio of that which AC has to EG; and it was demonstrated that it could not have that ratio to any solid less than the cone EFGHN: therefore the cone ABCDL has to the cone EFGHN, the triplicate ratio of that which AC has to EG. But as the cone is to the cone, so* the cylinder to the cylinder; for every cone is the third* part of the cylinder upon the same base, and of the same altitude: therefore also the cylinder has to the cylinder, the triplicate ratio of that which AC has to EG: wherefore similar cones, &c. Q. E. D.

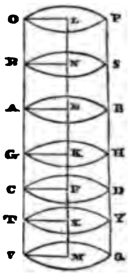
* 14. 5.

* 15. 5.

* 10. 12.

PROP. XIII. THEOR.

If a cylinder be cut by a plane parallel to its opposite planes, or bases: it divides the cylinder into two cylinders, one of which is to the other as the axis of the first to the axis of the other.



Let the cylinder AD be cut by the plane GH parallel to the opposite planes, AB, CD, meeting the axis EF in the point K, and let the line GH be the common section of the plane GH and the surface of the cylinder AD. Let AEFC be the parallelogram in any position of it, by the revolution of which about the straight line EF the cylinder AD is described; and let GK be the common section of the plane GH, and the plane AEFC. And because the parallel planes AB, GH are cut by the plane AEKG, AE, KG, their common sections
* 16. 11. with it are parallel;* wherefore AK is a parallelogram, and GK equal to EA the straight line from the centre of the circle AB: for the same reason, each of the straight lines drawn from the point K to the line GH may be proved to be equal to those which are drawn from the centre of the circle AB to its circumference, and are therefore all equal to one another: therefore
* 15 Def. 1. the line GH is the circumference of a circle * of which the centre is the point K; therefore the plane GH divides the cylinder AD into the cylinders AH, GD; for they are the same which would be described by the revolution of the parallelograms AK, GF, about the straight lines EK, KF: and it is to be shown that the cylinder AH is to the cylinder HC, as the axis EK to the axis KF.

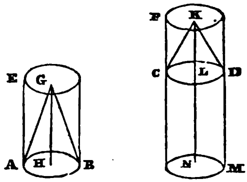
Produce the axis EF both ways; and take any number of straight lines EN, NL, each equal to EK; and any number FX, XM, each equal to FK; and let planes parallel to AB, CD pass through the points L, N, X, M; therefore the common sections of these planes with the cylinder produced are circles, the centres of which are the points L, N, X, M, as was proved of the plane GH; and these planes cut off the cylinders, PR, RB, DT, TQ. And because the axes LN, NE, EK are all equal: therefore the cylinders PR, RB, BG are* to one another as their bases; but * 11. 12.
their bases are equal, and therefore the cylinders PR, RB, BG are equal. And because the axes LN, NE, EK are equal to one another, as also the cylinders PR, RB, BG, and that there are as many axes as cylinders; therefore, whatever multiple the axis KL is of the axis KE, the same multiple is the cylinder PG of the cylinder GB. For the same reason whatever multiple the axis MK is of the axis KF, the same multiple is the cylinder QG of the cylinder GD: and if the axis KL be equal to the axis KM the cylinder PG is equal to the cylinder GQ; and if the axis KL be greater than the axis KM the cylinder PG is greater than the cylinder GQ; and if less, less. Since therefore there are four magnitudes, viz. the axes EK, KF, and the cylinders BG, GD, and that of the axis EK and cylinder BG there have been taken any equimultiples whatever, viz. the axis KL and cylinder PG; and of the axis KF and cylinder GD, any equimultiples whatever, viz. the axis KM and cylinder GQ; and since it has been demonstrated, if the axis KL be greater than the axis KM, the cylinder PG is greater than the cylinder GQ; and if equal, equal; and if less, less: therefore* the axis EK is to the axis KF, as * 5 Def. 5.
the cylinder BG to the cylinder GD. Wherefore, if a *cylinder, &c.* Q. E. D.

PROP. XIV. THEOR.

Cones and cylinders upon equal bases are to one another as their altitudes.

Let the cylinders EB, FD be upon the equal bases AB, CD: as the cylinder EB to the cylinder FD, so shall the axis GH be to the axis KL.

Produce the axis KL to the point N, and make LN equal to the axis GH, and let CM be a cylinder of which the base is CD, and axis LN, and because the cylinders EB, CM have the same altitude, they are to
* 11. 12. one another as their bases:* but their bases are equal, therefore also the cylinders EB, CM are equal. And



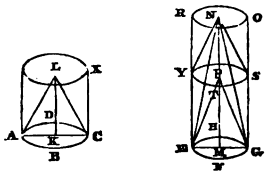
because the cylinder FM is cut by the plane CD parallel to its opposite planes, as the cylinder CM to the
* 13. 12. cylinder FD, so is* the axis LN to the axis KL; but the cylinder CM is equal to the cylinder EB, and the axis LN to the axis GH; therefore as the cylinder EB to the cylinder FD, so is the axis GH to the axis KL:
* 15. 5. and as the cylinder EB to the cylinder FD, so is* the cone ABG to the cone CDK, because the cylinders are
* 10. 12. triple* of the cones: therefore also the axis GH is to the axis KL, as the cone ABG to the cone CDK, and as the cylinder EB to the cylinder FD. Wherefore, cones, &c. Q. E. D.

PROP. XV. THEOR.

The bases and altitudes of equal cones and cylinders, are reciprocally proportional; and if the bases and altitudes be reciprocally proportional, the cones and cylinders are equal to one another.

Let the circles ABCD, EFGH, the diameters of which are AC, EG, be the bases, and KL, MN the axes, as also the altitudes, of equal cones and cylinders; and let ALC, ENG be the cones, and AX, EO the cylinders: the bases and altitudes of the cylinders AX, EO shall be reciprocally proportional: that is, as the base ABCD to the base EFGH, so shall the altitude MN be to the altitude KL.

Either the altitude MN is equal to the altitude KL, or these altitudes are not equal. First, let them be equal; and the cylinders AX, EO being also equal, and cones and cylinders of the same altitude being to one another as their bases,* therefore the base ABCD is equal* to the base EFGH; and as the base ABCD is to the base EFGH, so is the altitude MN to the altitude KL. * 11. 12. * A. 5.



But let the altitudes KL, MN, be unequal, and MN the greater of the two, and from MN take MP equal to KL, and through the point P cut the cylinder EO by the plane TYS parallel to the opposite planes of the circles EFGH, RO; therefore

the common section of the plane TYS and the cylinder EO is a circle, and consequently ES is a cylinder, the base of which is the circle EFGH, and altitude MP. And because the cylinder AX is equal to the cylinder
* 7. 5. EO, as AX is to the cylinder ES, so* is the cylinder EO to the same ES: but as the cylinder AX to the
* 11. 12. cylinder ES, so* is the base ABCD to the base EFGH; for the cylinders AX, ES are of the same altitude; and
* 13. 12. as the cylinder EO to the cylinder ES, so* is the altitude MN to the altitude MP, because the cylinder EO is cut by the plane TYS parallel to its opposite planes: therefore as the base ABCD to the base EFGH, so is the altitude MN to the altitude MP: but MP is equal to the altitude KL; wherefore as the base ABCD to the base EFGH, so is the altitude MN to the altitude KL; that is, the bases and altitudes of the equal cylinders AX, EO are reciprocally proportional.

But let the bases and altitudes of the cylinders AX, EO, be reciprocally proportional, viz. the base ABCD to the base EFGH, as the altitude MN to the altitude KL: the cylinder AX shall be equal to the cylinder EO.

First, let the base ABCD, be equal to the base EFGH; then because as the base ABCD is to the base EFGH, so is the altitude MN to the altitude KL:
* A. 5. therefore MN is equal* to KL, and therefore the
* 11. 12. cylinder AX is equal* to the cylinder EO.

But let the bases ABCD, EFGH be unequal, and let ABCD be the greater; and because, as the base ABCD is to the base EFGH, so is the altitude MN to
* A. 5. the altitude KL; therefore MN is greater* than KL. Then, the same construction being made as before, because as the base ABCD to the base EFGH, so is the altitude MN to the altitude KL; and because the altitude KL is equal to the altitude MP; therefore the
* 11. 12. base ABCD is* to the base EFGH, as the cylinder

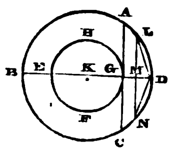
AX to the cylinder ES; and as the altitude MN to the altitude MP or KL, so is the cylinder EO to the cylinder ES: therefore the cylinder AX is to the cylinder ES, as the cylinder EO is to the same cylinder ES; whence the cylinder AX is equal to the cylinder EO: and the same reasoning holds in cones. Q. E. D.

PROP. XVI. PROB.

In the greater of two circles that have the same centre, to inscribe a polygon of an even number of equal sides, that shall not meet the lesser circle.

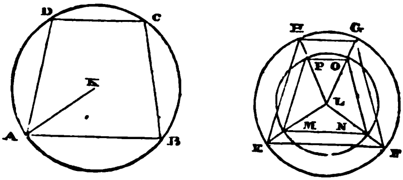
Let ABCD, EFGH be two given circles having the same centre K: it is required to inscribe in the greater circle ABCD a polygon of an even number of equal sides, that shall not meet the lesser circle.

Through the centre K draw the straight line BD, and from the point G, where it meets the circumference of the lesser circle, draw GA at right angles to BD, and produce it to C; therefore AC touches* the circle EFGH. Then, if the circumference BAD be bisected, and the half of it be again bisected, and so on, there must at length remain a circumference * 16. 3.



less* than AD: let this be LD: and from the point L draw LM perpendicular to BD, and produce it to N; and join LD, DN; therefore LD is equal to DN; and because LN is parallel to AC and that AC touches the circle EFGH; therefore LN does not meet the *Lemma

than, EF, FG, GH, HE. Then because EH is greater than MP, AD is greater than MP; and the circles ABCD, MNOP are equal; therefore the circumference



AD is greater than MP; for the same reason, the circumference BC is greater than NO; and because the straight line AB is greater than EF, which is greater than MN, much more is AB greater than MN: therefore the circumference AB is greater than MN; and, for the same reason, the circumference DC is greater than PO: therefore the whole circumference ABCD is greater than the whole MNOP; but this is impossible, because it is likewise equal to it: therefore KA is not less than LE; nor is it equal to it; therefore the straight line KA must be greater than LE. Q. E. D.

Cor. And if there be an isosceles triangle, the sides of which are equal to AD, BC, but its base less than AB, the greater of the two sides AB, DC; the straight line KA may, in the same manner, be demonstrated to be greater than the straight line drawn from the centre to the circumference of the circle described about the triangle.

PROP. XVII. PROB.

In the greater of two spheres which have the same centre, to inscribe a polyhedron, the superficies of which shall not meet the lesser sphere.

Let there be two spheres about the same centre A; it is required to inscribe in the greater a polyhedron, the superficies of which shall not meet the lesser sphere.

Let the spheres be cut by a plane passing through the centre; the common sections of it with the spheres shall be circles; because the sphere is described by the revolution of a semicircle about the diameter remaining unmoveable; so that in whatever position the semicircle be conceived, the common section of the plane in which it is with the superficies of the sphere is the circumference of a circle: and this is a great circle of the sphere, because the diameter of the sphere,
* 15. 3. which is likewise the diameter of the circle, is greater* than any straight line in the circle or sphere. Let then the circle made by the section of the plane with the greater sphere be BCDE, and with the lesser sphere be FGH; and draw the two diameters BD, CE, at right angles to one another; and in BCDE, the
* 16. 12. greater of the two circles, inscribe* a polygon of an even number of equal sides not meeting the lesser circle FGH; and let its sides, in BE the fourth part of the circle, be BK, KL, LM, ME; join KA and
* 12. 11. produce it to N; and from A draw* AX at right angles to the plane of the circle BCDE meeting the superficies of the sphere in the point X; and let planes pass through AX and each of the straight lines BD, KN, which, from what has been said, shall produce great circles on the superficies of the sphere, and let BXD, KXN be the semicircles thus made upon the diameters BD, KN; therefore, because XA is at right angles to the plane of the circle BCDE, every plane which
* 18. 11. passes through XA is at right* angles to the plane of the circle BCDE; wherefore the semicircles BXD, KXN are at right angles to that plane. And because the semicircles BED, BXD, KXN, upon the equal diameters BD, KN, are equal to one another their

halves BE, BX, KX, are equal to one another: therefore, as many sides of the polygon as are in BE, so many there are in BX, KX equal to the sides BK, KL, LM, ME. Let these polygons be described, and their sides be BO, OP, PR, RX; KS, ST, TY, YX, and join OS, PT, RY; and from the points O, S draw OV, SQ perpendiculars to AB, AK: and because the plane BOXD is at right angles to the plane BCDE, and in one of them BOXD, OV is drawn perpendicular to AB the common section of the planes, therefore OV is perpendicular* to the plane BCDE. For the same reason SQ is perpendicular to the same plane, because the plane KSXN is at right angles to the plane BCDE. * 4 Def. 11.

Join VQ; and because in the equal semicircles BXD, KXN the circumferences BO, KS are equal, and OV, SQ are perpendicular to their diameters, therefore* OV is equal to SQ, and BV equal to KQ: but the whole BA is equal to the whole KA, therefore the remainder VA is equal to the remainder QA: therefore as BV is to VA, so is KQ to QA, wherefore VQ is parallel* to BK. And because OV, SQ are each of them at right angles to the plane of the circle BCDE, * 26. 1. * 2. 6.

* 6. 11. OV is parallel* to SQ; and it has been proved that it
* 33. 1. is also equal to it; therefore QV, SO are equal* and parallel: and because QV is parallel to SO, and also
* 9. 11. to KB; OS is parallel* to BK; and therefore BO, KS which join them are in the same plane in which these parallels are, and the quadrilateral figure KBOS is in one plane. And if PB, TK be joined, and perpendiculars be drawn from the points P, T to the straight lines AB, AK, it may be demonstrated that TP is parallel to KB in the very same way that SO was
* 9. 11. shown to be parallel to the same KB: wherefore* TP is parallel to SO, and the quadrilateral figure SOPT is in one plane; for the same reason the quadrilateral
* 2. 11. TPRY is in one plane: and the figure YRX* is also in one plane; therefore, if from the points, O, S, P, T, R, Y there be drawn straight lines to the point A, there shall be formed a polyhedron between the circumferences BX, KX, composed of pyramids the bases of which are the quadrilaterals KBOS, SOPT, TPRY, and the triangle YRX, and of which the common vertex is the point A. And if the same construction be made upon each of the sides KL, LM, ME, as has been done upon BK, and the like be done also in the other three quadrants, and in the other hemisphere; there shall be formed a polyhedron inscribed in the sphere, composed of pyramids, the bases of which are the aforesaid quadrilateral figures, and the triangle YRX, and those formed in the like manner in the rest of the sphere, the common vertex of them all being the point A.

And the superficies of this polyhedron shall not meet the lesser sphere in which is the circle FGH. For, from
* 11. 11. the point A draw* AZ perpendicular to the plane of the quadrilateral KBOS, meeting it in Z, and join BZ, ZK. And because AZ is perpendicular to the plane KBOS, it makes right angles with every straight line

meeting it in that plane; therefore AZ is perpendicular to BZ and ZK: and because AB is equal to AK, and that the squares of AZ, ZB, are equal to the square of AB; and the squares of AZ, ZK, to the square of AK:* therefore the squares of AZ, ZB, are equal to the squares of AZ, ZK. Take from these equals the square of AZ, the remaining square of BZ is equal to the remaining square of ZK; and therefore the straight line BZ is equal to ZK. In like manner it may be demonstrated, that the straight lines drawn from the point Z to the points O, S are equal to BZ or ZK: therefore the circle described from the centre Z, and distance ZB, will pass through the points K, O, S, and KBOS will be a quadrilateral figure in the circle. And because KB is greater than QV, and QV equal to SO, therefore KB is greater than SO: but KB is equal to each of the straight lines BO, KS; wherefore each of the circumferences cut off by KB, BO, KS is greater than that cut off by OS; and these three circumferences, together with a fourth equal to one of them, are greater than the same three together with that cut off by OS; that is, than the whole circumference of the circle; therefore the circumference subtended by KB is greater than the fourth part of the whole circumference of the circle KBOS, and consequently the angle BZK at the centre is greater than a right angle. And because the angle BZK is obtuse, the square of BK is greater* than the squares of BZ, ZK; that is, greater than twice the square of BZ. Join KV, and because in the triangles KBV, OBV, KB, BV are equal to OB, BV, and that they contain equal angles; the angle KVB is equal* to the angle OVB: and OVB is a right angle; therefore also KVB is a right angle: and because BD is less than twice DV, *the rectangle* contained by DB, BV is less than *twice the rectangle* DV, VB: that is,* the square of

* 47. 1.

* 12. 2.

* 4. 1.

* 8. 6

KB is less than twice the square of KV: but the square of KB is greater than twice the square of BZ; therefore the square of KV is greater than the square of BZ: and because BA is equal to AK, and that the squares of BZ, ZA are equal together to the square of BA, and the squares of KV, VA to the square of AK; therefore the squares of BZ, ZA are equal to the squares of KV, VA; and of these the square of KV is greater than the square of BZ; therefore the square of VA is less than the square of ZA, and the straight line AZ greater than VA. Much more then is AZ greater than AG; because, in the preceding proposition, it was proved that KV falls without the circle FGH: and AZ is perpendicular to the plane KBOS, and is therefore the shortest of all the straight lines that can be drawn from A, the centre of the sphere to that plane: therefore the plane KBOS does not meet the lesser sphere.

And that the other planes between the quadrants BX, KX fall without the lesser sphere, is thus demonstrated. From the point A, draw AI perpendicular to the plane of the quadrilateral SOPT, and join IO; and, as was demonstrated of the plane KBOS and the point Z, in the same way it may be shown that the point I is the centre of a circle described about SOPT: and that OS is greater than PT; and PT was shown to be parallel to OS: therefore, because the two trapeziums KBOS, SOPT inscribed in circles, have their sides BK, OS parallel, as also OS, PT; and their other sides BO, KS, OP, ST all equal to one another, and that BK is greater than OS, and OS greater than
* 2 Lem. 12. PT; therefore the straight line ZB is greater* than IO. Join AO which will be equal to AB; and because AIO, AZB are right angles, the squares of AI, IO are equal to the square of AO, or of AB; that is, to the squares of AZ, ZB; and the square of ZB is greater

than the square of IO, therefore the square of AZ is less than the square of AI; and the straight line AZ less than the straight line AI: and it was proved that AZ is greater than AG; much more then is AI greater than AG: therefore the plane SOPT falls wholly without the lesser sphere. In the same manner it may be demonstrated that the plane TPRY falls without the same sphere, as also the triangle YRX, viz. by the Cor. of Lemma ii. And after the same way it may be demonstrated that all the planes which contain the polyhedron, fall without the lesser sphere: therefore in the greater of two spheres which have the same centre, a polyhedron is inscribed, the superficies of which does not meet the lesser sphere. Which was to be done.

But the straight line AZ may be demonstrated to be greater than AG otherwise, and in a shorter manner, without the help of Prop. xvi. as follows. From the point G, draw GU at right angles to AG and join AU. If then the circumference BE be bisected, and its half again bisected, and so on, there will at length be left a circumference less than the circumference which is subtended by a straight line equal to GU inscribed in the circle BCDE. Let this be the circumference KB: therefore the straight line KB is less than GU: and because the angle BZK is obtuse, as was proved in the preceding, therefore BK is greater than BZ: but GU is greater than BK; much more then is GU greater than BZ, and the square of GU than the square of BZ; and AU is equal to AB: therefore the square of AU, that is, the squares of AG, GU are equal to the square of AB, that is, to the squares of AZ, ZB; but the square of BZ is less than the square of GU; therefore the square of AZ is greater than the square of AG, *and the straight* line AZ consequently greater than *he straight line* AG.

COR. And if in the lesser sphere there be inscribed a polyhedron by drawing straight lines betwixt the points in which the straight lines from the centre of the sphere drawn to all the angles of the polyhedron in the greater sphere meet the superficies of the lesser; in the same order in which are joined the points in which the same lines from the centre meet the superficies of the greater sphere: the polyhedron in the sphere BCDE has to this other polyhedron the triplicate ratio of that which the diameter of the sphere BCDE has to the diameter of the other sphere. For if these two solids be divided into the same number of pyramids, and in the same order; the pyramids shall be similar to one another, each to each: because they have the solid angles at their common vertex, the centre of the sphere, the same in each pyramid, and their other solid angles at the bases equal to one another, each to each,* because they are contained by three plane angles each equal to each; and the pyramids are contained by the same number of similar planes; and are therefore similar* to one another, each to each: but similar pyramids have to one another the triplicate* ratio of their homologous sides. Therefore the pyramid of which the base is the quadrilateral KBOS, and vertex A, has to the pyramid in the other sphere of the same order, the triplicate ratio of their homologous sides; that is, of that ratio, which AB from the centre of the greater sphere has to the straight line from the same centre to the superficies of the lesser sphere. And in like manner, each pyramid in the greater sphere has to each of the same order in the lesser, the triplicate ratio of that which AB has to the semidiameter of the lesser sphere. And as one antecedent is to its consequent, so are all the antecedents to all the consequents: wherefore the whole polyhedron in the greater sphere has to the whole poly-

* B. 11.

* 11 Def. 11.

* Cor. 8. 12.

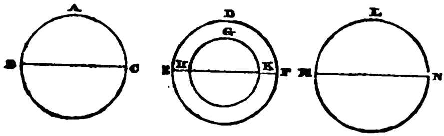
hedron in the other, the triplicate ratio of that which AB the semidiameter of the first has to the semidiameter of the other; that is, which the diameter BD of the greater has to the diameter of the other sphere.

PROP. XVIII. THEOR.

Spheres have to one another the triplicate ratio of that which their diameters have.

Let ABC, DEF be two spheres of which the diameters are BC, EF: the sphere ABC shall have to the sphere DEF the triplicate ratio of that which BC has to EF.

For, if it has not, the sphere ABC must have to a sphere either less or greater than DEF, the triplicate ratio of that which BC has to EF. First, if possible, let it have that ratio to a less, viz. to the sphere GHK; and let the sphere DEF have the same centre with GHK; and in the greater sphere DEF inscribe* a polyhedron, the superficies of which does not meet the * 17. 12.



lesser sphere GHK; and in the sphere ABC inscribe another similar to that in the sphere DEF: therefore the polyhedron in the sphere ABC has to the polyhedron in the sphere DEF, the triplicate ratio* of that which BC has to EF. But the sphere ABC has to the *Cor. 17. 12.
sphere GHK, the triplicate ratio of that which BC has to EF; therefore, as the sphere ABC to the sphere GHK, so is the polyhedron in the sphere ABC to the polyhedron in the sphere DEF. But the sphere ABC

* 14. 5. is greater than the polyhedron in it; therefore* also the sphere GHK is greater than the polyhedron in the sphere DEF: but this is impossible, because it is less, being contained within it: therefore the sphere ABC has not to any sphere less than DEF, the triplicate ratio of that which BC has to EF. In the same manner, it may be demonstrated, that the sphere DEF has not to any sphere less than ABC, the triplicate ratio of that which EF has to BC. Nor can the sphere ABC have to any sphere greater than DEF, the triplicate ratio of that which BC has to EF. For, if it can, let it have that ratio to a greater sphere LMN: therefore, by inversion, the sphere LMN has to the sphere ABC, the triplicate ratio of that which the diameter EF has to the diameter BC. But as the sphere LMN to ABC, so is the sphere DEF to some sphere, which must be
* 14. 5. less* than the sphere ABC, because the sphere LMN is greater than the sphere DEF: therefore the sphere DEF has to a sphere less than ABC the triplicate ratio of that which EF has to BC; which was shown to be impossible: therefore the sphere ABC has not to any sphere greater than DEF the triplicate ratio of that which BC has to EF: and it was demonstrated, that neither has it that ratio to any sphere less than DEF: therefore the sphere ABC has to the sphere DEF, the triplicate ratio of that which BC has to EF. Q. E. D.

END OF THE ELEMENTS.

APPENDIX.

APPENDIX.

MISCELLANEOUS EXERCISES

IN

PLANE GEOMETRY.

1. From two given points to draw two straight lines which shall be equal to one another, and which shall meet in the same point in a line given in position.

2. From two given points, on the same side, or opposite sides of a line given in position to draw two lines, which shall meet in that line, and make equal angles with it.

3. The two sides of any triangle are together greater than the double of the straight line which joins the vertex, and the point of bisection of the base.

4. To trisect a given finite straight line.

5. The difference between two sides of a triangle is less than the third side.

6. The three straight lines drawn from the points of bisection of the three sides to the opposite angles, pass through the same point.

7. The straight line joining the points of bisection of two sides of a triangle, is parallel to the base.

8. Straight lines joining the successive middle points of the sides of a quadrilateral figure, form a parallelogram.

9. In any triangle, the sum of the squares of the sides is equal to twice the square of half the base, and twice the square of the straight line, which joins the point of bisection with the vertex.

10. Four times the sum of the squares of the three straight lines, drawn from the angles of a triangle, to the points of bisection of the opposite sides, are equal to three times the sum of the squares of the three sides.

11. The squares of the straight lines drawn from any point to two opposite angles of a rectangle, are together equal to the squares of those drawn to the other angles.

12. If the sides of an equilateral and equiangular pentagon be produced to meet, the angles formed by these lines are together equal to two right angles.

13. If the sides of an equilateral and equiangular hexagon be produced to meet, the angles formed by these lines, are together equal to four right angles.

14. If from the extremities of the diameter of a semicircle, perpendiculars be let fall upon any line cutting the semicircle, the parts intercepted between the perpendiculars and the circumference are equal.

15. If on each side of any point in the circumference of a circle, any number of equal arcs be taken, and the extremities of each pair joined, the sum of the chords so drawn will be equal to the last chord produced to meet a line drawn from the given point through the extremity of the first arc.

16. If one circle touch another externally or internally, any straight line drawn through the point of contact will cut off similar segments.

17. From two given points in the circumference of a given circle, to draw two lines to a point in the circumference, which shall cut a line given in position, so that the part of it intercepted by them may be equal to a given line.

18. If two circles touch each other, and also touch a straight line, the part of the line between the points of contact is a mean proportional between the diameters of the circles.

19. If from any point within an equilateral triangle, perpendiculars be drawn to the sides, they are together equal to a perpendicular from any of the angles to the opposite side.

20. The three straight lines which bisect the angles of a triangle, meet in the same point.

21. If from the angles of a triangle, perpendiculars be drawn to the opposite sides, they will pass through the same point.

22. Within or without a triangle, to draw a straight line parallel to the base, such that it may be equal to the parts of the other sides, or of their continuations, intercepted between it and the base.

23. If the circumference of a circle be cut by two straight lines which are perpendicular to one another, the squares of the four segments between the point of intersection of the two lines, and the points in which they meet the circumference, are together equal to the square of the diameter.

24. Describe a circle touching a given straight line, and also passing through two given points.

25. In an isosceles triangle to inscribe three circles touching each other, and each touching two of the three sides of the triangle.

26. In a given triangle to inscribe a rectangle ot a given magnitude.

27. In a given triangle to inscribe a rectangle which shall be similar to a given rectangle.

28. In a given triangle to inscribe a square.

29. If in proceeding round a square in the same *direction, points* be taken in the sides, or the sides *produced* at equal distances from the four angles, the

figure contained by the straight lines which join them shall also be a square.

30. If squares be described on the three sides of a right-angled triangle, and the extremities of the adjacent sides be joined, the triangles thus formed are equal to the given triangle and to each other.

31. If squares be described on the hypothenuse and sides of a right-angled triangle, and the extremities of the sides of the former, and the adjacent sides of the others be joined, the sum of the squares of the lines joining them will be equal to five times the square of the hypothenuse.

32. The squares of the four sides of a quadrilateral figure are together equal to the squares of the diagonals, together with four times the square of the straight line joining their middle points.

33. To bisect a triangle by a line drawn parallel to one of its sides.

34. If in a right-angled triangle a perpendicular be drawn from the right angle to the hypothenuse, and circles inscribed in the triangles on each side of it, their diameters will be to each other as the subtending sides of the right-angled triangle.

35. It is required to inscribe a square in a given semicircle.

36. Draw straight lines across the angles of a given square, so as to form an equilateral and equiangular octagon.

37. If on one side of an equilateral triangle as a diameter, a semicircle be described, and from the opposite angle two straight lines be drawn to trisect that side, these lines produced will also trisect the semicircumference.

38. The square of the side of an equilateral triangle inscribed in a circle, is equal to three times the square of the radius.

39. To draw straight lines from the extremities of the chord, to a point in the circumference of the circle, so that their sum may be equal to a given straight line.

40. Given the perimeter of a right-angled triangle, and the perpendicular from the right angle upon the hypothenuse, to construct the triangle.

41. To describe an isosceles triangle, having the angle at the vertex triple of each of the angles at the base.

42. If on the three sides of any triangle, equilateral triangles be described, either all externally, or all internally; the straight lines which join the centres of the circles inscribed in those three triangles will form an equilateral triangle.

43. To divide a given straight line, so that the rectangle contained by the segments shall be equal to a given rectangle, not exceeding the square upon half the given line.

44. To produce a given straight line, so that the rectangle contained by the whole line thus produced, and the part produced, shall be equal to a given rectangle.

45. In a given circle to inscribe three equal circles, touching each other and the given circle.

46. If the tangents drawn to every two of three unequal circles be produced till they meet, the points of intersection will be in the same straight line.

47. In the given straight line joining the centres of two given unequal circles, that lie wholly without each other, to find a point such, that the two tangents drawn from it to the circles, shall be equal to one another.

48. To find a point from which, if straight lines be drawn to three given points, they will be proportional to three given straight lines.

49. Each of the complements of the parallelograms about the diagonal of a parallelogram, is a mean proportional between these parallelograms.

50. To bisect a given rectangle, by two straight lines

drawn parallel to two of its adjacent sides, and equally distant from them.

51. The two angles which straight lines drawn from three given points to a fourth point being given: to find the position of that point.

52. Given the base of a triangle, the vertical angle and the rectangle of the sides: to construct the triangle.

53. If from the extremities of the base of any triangle, two straight lines be drawn intersecting each other in the perpendicular, and terminating in the opposite sides; straight lines drawn from thence to the intersection of the perpendicular with the base, will make equal angles with the base.

54. If from any angle of a rectangular parallelogram, a line be drawn to the opposite side, and from the adjacent angle of the trapezium thus formed, another be drawn perpendicular to the former; the rectangle contained by these two lines is equal to the given parallelogram.

55. If through any point within a triangle, three lines be drawn parallel to the sides, the solids formed by the alternate segments of these lines are equal.

56. To describe an equilateral triangle, equal to a given isosceles triangle.

57. To describe a circle which shall touch a straight line in a given point, and also touch a given circle.

58. If on the side of a rectangular parallelogram as a diameter, a semicircle be described, and from any point in the circumference lines be drawn through its extremities, to meet the opposite side produced; the altitude of the parallelogram is a mean proportional between the segments cut off.

59. If a straight line be divided into any two parts. and upon the whole and the two parts semicircles be described, and from the point of section a perpendicular be drawn, on each side of which circles are de-

scribed touching it and the semicircles, these circles will be equal.

60. Given the lengths of three lines drawn from the angles to the points of bisection of the opposite sides: to construct the triangle.

61. Given the vertical angle, the difference of the two sides containing it, and the difference of the segments of the base made by a perpendicular from the vertex: to construct the triangle.

62. Given the three distances of a point within a square from three angular points: to construct the square.

63. To divide a circle into any proposed number of equal parts, by means of concentric circles.

64. To divide a triangle into three equal parts, by lines drawn from a given point within it.

65. To divide a quadrilateral into three equal parts, by lines drawn from the vertex of one of its angles.

66. To divide an irregular pentagon into three equal parts, by lines drawn from the vertex of one of the angles.

67. The sum of the perpendiculars drawn from the centre of the circle described about a triangle to the three sides, is equal to the sum of the radii of the circumscribed and inscribed circles.

68. Given the three lines drawn from the vertex of a triangle to the base, one of them perpendicular to the base, one bisecting the vertical angle, and one bisecting the base: to construct the triangle.

69. Given the base of a triangle, the vertical angle, and the straight line bisecting that angle: to construct the triangle.

70. The segments of the base made by a perpendicular from the vertical angle, and the ratio of the sides are *given: to* construct the triangle.

CRITICAL QUESTIONS

ON THE

ELEMENTS OF EUCLID.

1. What is *geometry?*

Geometry is the science of magnitude, or local extension, and the subjects which it considers are extent of distance, extent of surface, and extent of capacity, or solid content.

2. What is the true *foundation* of every system of instruction?

Adequate and precise definitions.

3. What is a *definition?*

It is a clear explanation of the meaning of any term or word employed, and distinguishes, with absolute precision, the idea expressed by that term, from the idea expressed by any other term.

4. Define a *postulate.*

A postulate is the demand of the author, that the reader may admit the possibility of performing some specified operation.

5. Define an *axiom.*

An axiom is a self-evident truth.

6. What do you understand by a *proposition?*

It is a truth proposed to be proved, or an operation required to be done; and is either a theorem or a problem.

7. What is meant by a *theorem?*

A theorem is a truth proposed to be demonstrated, or whose evidence depends upon a train of reasoning.

8. What is meant by a *problem?*

A problem is an operation required to be performed, such as the drawing of a line, or the construction of a figure.

9. What is a proposition named, which is preparatory to one or more others, and which is of no other use?

A *lemma*.

10. What do you understand by the term *demonstration?*

It is the reasoning by which a truth is established.

11. What is a *corollary?*

It is an inference which arises easily and immediately from some other principle, not requiring any lengthened process of reasoning to establish its truth.

12. What is the meaning of a *scholium?*

A scholium is a remark subjoined to a demonstration.

13. What is the meaning of the term *hypothesis?*

An hypothesis is a supposition, which may be either true or false.

14. What is a *direct* demonstration?

A direct demonstration commences with admitted or demonstrated truths, and proceeds by deducing a series of other truths, each depending on the preceding, until it finally arrives at the truth proposed to be proved.

15. What is an *indirect* demonstration?

An indirect or *negative* demonstration consists in assuming as true a proposition which directly contradicts the one we mean to prove; and on this assumption a demonstration is founded, which issues in a result contrary to a self-evident or demonstrated truth; thus proving the truth of the proposition, by demonstrating that the supposition of its contrary leads to an absurd conclusion.

16. When are two propositions contrary to one another?

When the one affirms what the other denies, or

denies what it affirms; thus, if it be affirmed that three and five are eight, the contrary proposition is, that three and five are *not* eight.

17. When are two propositions *converse* to one another?

When, in the language of logic, the *subject* of one is made the *predicate* of the other, and *vice versâ*.

18. What is defective in Euclid's definition of a point?

It includes no positive property of a point; because it does not define what a point is, but what it is not.

19. Give a more correct definition of a point.

A point is that which has position, but not magnitude.

20. In geometrical investigations, physical lines and points are used instead of mathematical ones; is the reasoning vitiated on this account?

No: because the reasoning is conducted on the supposition that the point has no magnitude, and the line no breadth.

21. What axiom is not given by Euclid in a distinct form, though tacitly employed by him?

The whole is equal to all its parts taken together.

22. Does the magnitude of an angle depend on the length of the lines which contain it, or on what?

Simply on their relative position.

23. Is there any redundancy in the enunciation of Prop. i. Book i.?

Yes; the word *finite* is unnecessary, for a *given* straight line must be finite. Euclid, however, employs the word to show that the line is not of unlimited length, but is given in magnitude, as well as in position.

24. What proposition affords the first instance in which the indirect method of demonstration is introduced?

The sixth has usually been named as the first proposition in which the indirect mode of proof is em-

ployed; but the first instance is in the demonstration of Prop. iv.

25. Why is the phrase "on the side remote from A," added in this edition to the construction of Prop. ix. Book i.?

Because, if the triangle were described on the other side, its vertex might fall on the vertex of the given angle, and in this case the solution would evidently fail.

26. What proposition of Book i. may be omitted, and why?

The seventeenth; because it is included in the thirty-second of the same Book, and is not referred to in anything preceding that proposition.

27. What problem in Book i. is a particular case of Prop. ix. Book vi.?

The tenth.

28. How many sides has that polygon, the interior angles of which are together equal to twenty right angles?

Twelve sides.

29. What is deficient in the demonstration of Prop. xlviii. Book i.?

The squares upon equal straight lines are assumed equal, without demonstration.

30. From what proposition of Book i. may the squares of equal straight lines be inferred to be equal?

From the forty-sixth.

31. What is the principal object of the first Book of Euclid?

The principal object of the first Book, is to establish the truth of Prop. xlvii. which is so important in scientific inquiries, that little or no progress can be made in science without it.

32. Define a gnomon.

It is the part of a parallelogram which remains *when either of* the parallelograms about one of the *diagonals is* removed.

33. How much greater is the square of a line than the squares of its two parts?

Twice the rectangle contained by the two parts.

34. Of what does the second Book of Euclid treat?

Of right-angled parallelograms, and of squares.

35. Why is it a condition in Prop. iii. iv. Book iii. that the lines bisected must not pass through the centre.

Because if both lines pass through the centre, they must bisect each other, but may be inclined to each other at any angle.

36. Give a correct demonstration of Prop. ix. Book iii. without a diagram.

Since from any point which is *not* the centre of the circle, only *two equal straight* lines can be drawn to the circumference (vii. iii.); therefore the point from which more than two *equal* straight lines can be drawn to the circumference, cannot be any other than the centre. Dr. Simson's demonstration of this proposition is open to objection; because, by assuming different positions for the fictitious centre, the reasoning will require considerable modification. The following is the shortest proof:—" For if the point be *not* the centre, there could be but *two* equal straight lines drawn from it to the circumference, (vii. iii.); hence the truth of the proposition.

37. Prove the truth of Prop. xiii. Book iii. without a diagram.

If two circles touch one another, the straight line joining their centres passes through the point of contact (xi. and xii. iii.); therefore, if the circles could touch one another in more points than one, more straight lines than one could be drawn through their centres without coinciding with one another; which is impossible.

38. Is there any illegitimate assumption in Simson's demonstration of Prop. xx. Book iii.?

Yes: it takes for granted a particular case of Prop. v. Book v. *viz.*: "If a magnitude be double of another, and if a part taken from the first, be double of a part taken from the second, the remainder of the first is double of the remainder of the second."

39. What is defective in the demonstration of Prop. xxx. Book iii.?

In the demonstration it is asserted, that "equal straight lines cut off equal circumferences;" but it has never been proved, that in the *same* circle equal straight lines cut off equal circumferences.

40. Is there any defect in the demonstration of Prop. iii. Book iv.?

Yes: it is assumed that LM,MN, NL, meet two and two, and form a triangle LMN, without demonstration.

41. What useful theorem may be deduced from Prop. iv. Book iv.?

The diameter of the circle inscribed in a right-angled triangle is equal to the excess of the sum of the legs above the hypothenuse.

42. What other regular polygons, besides those of which the number of sides is some multiple of 3, 4, 5, may be constructed by elementary geometry?

All those regular polygons, the number of whose sides is a power of 2 increased by unity, and is a prime number, that is, a number not producible by the multiplication of any two whole numbers greater than unity; such as polygons of 17, 257, and 65,537 sides; for $17 = 2^4 + 1$, $257 = 2^8 + 1$, $65,537 = 2^{16} + 1$. The construction of a septemdecagon, or 17-sided polygon, has been the subject of some very elegant researches by Gauss, Lowry, and others.

43. Define the term *equimultiples.*

Equimultiples of magnitudes are multiples that contain *those* magnitudes respectively, the same number *of times.*

44. What is meant by the expression, "similitude of ratios," in Def. viii. Book v.?

It signifies simply the *equality* of ratios.

45. When are numbers or magnitudes said to be *commensurable*, or *incommensurable*?

According as they have a common measure or not.

46. When are magnitudes said to be of the *same kind*?

When they admit of comparison as to their equality or inequality.

47. What is meant by "continual proportionals?"

The meaning is, that each of the terms, except the first and last, is used as the consequent of one ratio, and the antecedent of another.

48. In continual proportionals, of what are the ratios of the first to the third, the first to the fourth, the first to the *n*th compounded?

Two equal ratios, three equal ratios, $n-1$ equal ratios: hence the terms duplicate ratio, triplicate ratio, and so on.

49. What is defective in the demonstration of Prop. i. Book vi.?

It is assumed, that if the base of one triangle be greater or less than the base of another of the same altitude, then the area of the former is greater or less than the area of the latter, without proof or axiom. Besides, it ought to have been proved that the triangles and parallelograms are to one another as their bases; whereas it is proved that the bases are to one another as the triangles and parallelograms.

50. What is defective in the enunciation of Prop. ii. Book vi.?

It does not specify in the analogy what lines are homologous to one another; but assumes that the segments between the parallel and the base are homologous to one another.

51. Enunciate those propositions of Book vi. which

peculiarly facilitate the application of algebra to geometrical investigations.

(1.) "The sides about the equal angles of equiangular triangles are proportionals," &c.

(2.) "If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means," &c.

52. Specify those propositions of Book vi. which generalize certain propositions in Book i.

Prop. iv. v. vi. Book vi., generalize Prop. xxvi. viii. iv. Book i.

53. Of what does the sixth Book of Euclid treat?

Of the sides and areas of certain rectilineal figures, and contains the investigation of lines that have a proposed ratio to given lines.

54. What do the eleventh and twelfth Books treat of?

Of the geometry of planes and solids.

55. Of how many Books do the Elements of Euclid consist?

Fifteen.

56. About what period did Euclid flourish?

About 300 years, B. C. at the time Ptolemy Lagos was king of Egypt.

57. What answer did Euclid give to the question of his pupil, King Ptolemy—"Is there no shorter way of coming at geometry than by your Elements?"

"There is no ROYAL road to GEOMETRY."

NOTES.

NOTE I.

ANOTHER DEMONSTRATION OF PROPOSITION XIII. BOOK II.

(*See the figures in p.* 64.)

Let ABC be any triangle, and the angle at B one of its acute angles; and upon BC, or its prolongation,
* 12. 1. one of the sides containing it, let fall the perpendicular* AD from the opposite angle: the square of the side AC, subtending the angle B, shall be less than the squares of AB, BC, by twice the rectangle CB, BD.

For if the perpendicular AD falls within the triangle ABC, the straight line BC is divided into two parts in the point D, and if it falls without the triangle, then BD is divided into two parts in the point C; therefore,
* 7. 2. in either case, the squares of CB, BD are equal* to twice the rectangle contained by CB, BD, together with the square of DC: to each of these equals add the square of AD; therefore, the squares of CB, BD, DA,
* 2. Ax. are equal* to twice the rectangle CB, BD, together with the squares of AD, DC; but the square of AB is
* 47. 1. equal* to the squares of BD, DA, because the angle BDA is a right angle; and the square of AC is equal to the squares of AD, DC; therefore, the squares of CB, BA are equal to the square AC, and twice the rectangle CB, BD; that is, the square of AC alone, is less than the squares of CB, BA, by twice the rectangle CB, BD. But if the side AC be perpendicular to BC, then BC is the straight line between the perpendicu-

and the acute angle at B: and because the square of AB is equal* to the squares of BC, CA; therefore the squares of AB, BC are equal to the square of AC, and twice the square of BC: therefore in every triangle &c. Q. E. D. * 47. 1.

NOTE II.

NUMERICAL EXPLANATION OF DEFINITION V. BOOK V.

1. *Let it be required to ascertain whether* 4 *has the same ratio to* 7 *which* 24 *has to* 42, *or if* 4 : 7 : : 24 : 42.

Of the *first* and *third* take any equimultiples 12 and 72; and of the *second* and *fourth* take any equimultiples 14 and 84; then the multiples of the four proposed numbers, taken in their order, are

12, 14, 72, 84.

Now the multiple of the *first* is less than that of the *second*, and the multiple of the *third* is less than that of the *fourth*.

Again, of the *first* and *third* take any equimultiples 28 and 168, and of the *second* and *fourth* take the equimultiples 28 and 168; then the multiples are

28, 28, 168, 168;

where the multiple of the *first* is equal to that of the *second*, and the multiple of the *third* is also equal to that of the *fourth*.

Lastly, of the *first* and *third* take any equimultiples 20 and 120, and of the *second* and *fourth* take equimultiples 14 and 84; then these multiples are

20, 14, 120, 84.

Here the multiple of the *first* is greater than that of the *second*, and the multiple of the *third* is also greater

than that of the *fourth;* therefore the three conditions enunciated in the definition are completely fulfilled and consequently

$$4 : 7 :: 24 : 42;$$

or 4 has to 7 the same ratio which 24 has to 42.

2. *Determine, by Euclid's definition of proportion, whether*

$$7 : 22 :: 113 : 355.$$

The failure of fulfilling any one of the three conditions stated in the definition, will be a certain indication that the four given numbers cannot constitute a proportion. Now, if by taking equimultiples of the first and third we have

154, 22, 2486, 355,

and by taking equimultiples of the second and fourth we have

154, 154, 2486, 2485;

then the multiple of the first is equal to that of the second, but the multiple of the third is *not* equal to that of the fourth; consequently the proposed numbers cannot constitute a proportion, because they fail to fulfil one of the conditions of the definition of proportion.

NOTE III.

SYNOPSIS OF PROPORTION.

If	a	:	b	::	c	:	d
Permutando	a	:	c	::	b	:	d
Invertendo	b	:	a	::	d	:	c
Componendo	$a+b$	:	b	::	$c+d$	:	d
Dividendo	$a-b$	:	b	::	$c-d$	:	d
Convertendo	a	:	$a-b$	::	c	:	$c-d$
or	a	:	$a+b$	::	c	:	$c+d$
Also, if	a	:	b	::	d	:	e
and	b	:	c	::	e	:	f
then *ex æquo ordinate*	a	:	c	::	d	:	f
But, if	a	:	b	::	e	:	f
and	b	:	c	::	d	:	e
then, *ex æquo perturbate*	a	:	c	::	d	:	f
Again, if	a	:	b	::	c	:	d
then	$a+b$	:	$a-b$	::	$c+d$	:	$c-d$
also	ma	:	mb	::	nc	:	nd
or	ma	:	nb	::	mc	:	nd
....	$\frac{a}{m}$	:	$\frac{b}{m}$	::	$\frac{c}{n}$	:	$\frac{d}{n}$
....	$\frac{a}{m}$	:	$\frac{b}{n}$	::	$\frac{c}{m}$	:	$\frac{d}{n}$
....	a^n	:	b^n	::	c^n	:	d^n
Also, if	a	:	b	::	c	:	d
and	e	:	f	::	g	:	h
and	k	:	l	::	m	:	n
then	ack	:	bfl	::	cgm	:	dhn

NOTE IV.

ON INCOMMENSURABLE MAGNITUDES.

1. *A magnitude that measures two other magnitudes, measures also their sum and difference.*

Let M be a magnitude that measures each of the magnitudes A and B; and let M measure A, *m* times, and B, *n* times; then A is equal to *m* times M, and B is equal to *n* times M; therefore A and B together are equal to *m* times M, together with *n* times M; that is, to as many times M as there are units contained in the sum of *m* and *n*; hence M measures the sum of A and B. It likewise measures their difference; for the difference between A and B is equal to the difference between *m* times M and *n* times M; that is to as many times M as there are units in the difference between *m* and *n*; hence M measures the difference of A and B.

Hence if M measures B, and either the sum of A and B, or the excess of A above B, it will measure A. For if M measure B and the excess of A above B, it will measure A, which is the sum of B, and the excess of it above B. For the same reason, if M measures B and the sum of A and B, it will measure their difference; that is, M measures A.

2. *Two magnitudes of the same kind being given, to find their greatest common measure.*

Let A and B be the given magnitudes; it is required to find the greatest magnitude that will measure each of them. Let A be the greater of the two magnitudes, and find the multiple of B that is nearest to A, either greater or less than it, and let the *difference* between A and this multiple be C, which must be either equal

to the half of B, or *less* than its half. In like manner find the multiple of C nearest to B, and let the difference between B and this multiple be D, a magnitude either *less* than the half of C, or exactly equal to its half: proceed in this manner till no remainder be left, or till the last *difference* measures the preceding one; then shall the last *difference* be the greatest common measure required.

$$\begin{array}{llll}
B)\,A\,(p & & & \\
\;\;\underline{pB} & & & \\
\;\;\;\;C)\,B\,(q & & & \\
\;\;\;\;\;\;\underline{qC} & & & \\
\;\;\;\;\;\;\;\;D)\,C\,(r & & & \\
\;\;\;\;\;\;\;\;\;\;\underline{rd} & & & \\
\;\;\;\;\;\;\;\;\;\;\overline{} & & &
\end{array}$$

Let D be the last *difference*; then D measures C, and therefore it measures any multiple of C; hence D measures both the sum and difference of D, and a multiple of C; but the sum or difference of D and a multiple of C is equal to B; hence D measures B, and therefore D measures any multiple of B; but the difference of A and a multiple of B is C, which is measured by D; hence D measures the sum or difference of C and a multiple of B; that is, D measures A; consequently D measures both A and B.

Also, D is the *greatest* common measure of A and B; for since the magnitude required measures A, B, and also any multiple of B; therefore it measures the difference between A and a multiple of B; that is, it measures C. Again, since it must measure any multiple of C, it must also measure the difference between B and a multiple of C; that is, it measures D.

But a magnitude cannot be measured by a magnitude greater than itself; hence the last difference is the *greatest* common measure of A and B.

If the preceding process is interminable, the given magnitudes cannot have a common measure, and are consequently *incommensurable*.

THE END.

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